The λ-calculus: typed and untyped

https://cl-teaching.uibk.ac.at/26w/lc/

Solutions to selected exercises from Lecture 3

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Assessed exercises from Lecture 3

(https://cl-teaching.uibk.ac.at/26w/lc/exercises/03.pdf)

Exercise 1 (2 points)

Given the λ-term $t = \lambda f^{X \Rightarrow X}.\lambda x^{X}.f(fx)$ in Church-style, find two *distinct* derivations $\mathcal{D} \triangleright_{\mathsf{STLC}_{\mathrm{Church}}} \Gamma \vdash t : A$ and $\mathcal{D}' \triangleright_{\mathsf{STLC}_{\mathrm{Church}}} \Gamma' \vdash t : A'$ for some environments Γ, Γ' and some types A, A'.

Do they contradict the proposition about the uniqueness of type and derivation in $\mathsf{STLC}_{\mathrm{Church}}$ seen in Lecture 3, p. 19? Justify briefly your answer.

Hint. The derivations $\mathcal{D}$ and $\mathcal{D}'$ must be distinct, but possibly $\Gamma = \Gamma'$ and/or $A = A'$.

Solution to Exercise 1

Two distinct derivations $\mathcal{D}$ and $\mathcal{D}'$ typing $\lambda f^{X \Rightarrow X}.\lambda x^{X}.f(fx)$ in $\mathsf{STLC}_{\mathrm{Church}}$ are below.

$$\mathcal{D} = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{f : X \Rightarrow X,\, x : X \vdash f : X \Rightarrow X}\,\mathsf{var} \quad \dfrac{\dfrac{}{f : X \Rightarrow X,\, x : X \vdash f : X \Rightarrow X}\,\mathsf{var} \quad \dfrac{}{f : X \Rightarrow X,\, x : X \vdash x : X}\,\mathsf{var}}{f : X \Rightarrow X,\, x : X \vdash fx : X}\,@}{f : X \Rightarrow X,\, x : X \vdash f(fx) : X}\,@}{f : X \Rightarrow X \vdash \lambda x^{X}.f(fx) : X \Rightarrow X}\,\lambda}{\vdash \lambda f^{X \Rightarrow X}.\lambda x^{X}.f(fx) : (X \Rightarrow X) \Rightarrow X \Rightarrow X}\,\lambda$$

$$\mathcal{D}' = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y{:}Y,\, f{:}X \Rightarrow X,\, x{:}X \vdash f : X \Rightarrow X}\,\mathsf{var} \quad \dfrac{\dfrac{}{y{:}Y,\, f{:}X \Rightarrow X,\, x{:}X \vdash f : X \Rightarrow X}\,\mathsf{var} \quad \dfrac{}{y{:}Y,\, f{:}X \Rightarrow X,\, x{:}X \vdash x : X}\,\mathsf{var}}{y{:}Y,\, f{:}X \Rightarrow X,\, x{:}X \vdash fx : X}\,@}{y{:}Y,\, f{:}X \Rightarrow X,\, x{:}X \vdash f(fx) : X}\,@}{y{:}Y,\, f : X \Rightarrow X \vdash \lambda x^{X}.f(fx) : X \Rightarrow X}\,\lambda}{y{:}Y \vdash \lambda f^{X \Rightarrow X}.\lambda x^{X}.f(fx) : (X \Rightarrow X) \Rightarrow X \Rightarrow X}\,\lambda$$

Derivations $\mathcal{D}$ and $\mathcal{D}'$ are distinct because they have different environments: it is empty in the conclusion of $\mathcal{D}$, while it is $y : Y$ in the conclusion of $\mathcal{D}'$. The difference in the environments of their conclusions means that $\mathcal{D}$ and $\mathcal{D}'$ do not fulfill the hypotheses of the proposition about the uniqueness of type and derivation in $\mathsf{STLC}_{\mathrm{Church}}$ in Lecture 3, p. 19, hence there is no contradiction (that is, the proposition is vacuously true with $\mathcal{D}$ and $\mathcal{D}'$ above).

Exercise 2 (4 points)

In the saturation lemma for $\mathsf{STLC}_{\mathrm{Curry}}$ seen in Lecture 3, p. 28, prove Point 2 when the type B is X (a ground type).

Hint. The solution to a very similar non-assessed exercise from Lecture 3 may be useful (if used, its proof should be part of the submission).

Solution to Exercise 2

We have to prove that if $\Gamma \vdash_{\text{Curry}} xt_1 \dots t_n : X$ and $t_1, \dots, t_n$ are SN (i.e. strongly normalizing), then $\langle \Gamma; xt_1 \dots t_n \rangle \in \mathsf{Red}_X$. Since X is a ground type, according to the definition of Red_X we show that:

1. $xt_1 \dots t_n$ is SN, and
2. $\Gamma \vdash_{\text{Curry}} xt_1 \dots t_n : X$.

Point 1 holds thanks to Exercise 6.1 below; Point 2 holds by hypothesis.

Exercise 3 (4 points)

In the saturation lemma for $\mathsf{STLC}_{\text{Church}}$ seen in Lecture 3, p. 28, prove Point 3 when the type B is X (a ground type).

Hint. The solution to a very similar non-assessed exercise from Lecture 3 may be useful (if used, its proof should be part of the submission).

Solution to Exercise 3

We have to prove that if $\langle \Gamma; s\{u/x\}t_1 \dots t_n \rangle \in \mathsf{Red}_X$ and $\Gamma \vdash_{\text{Church}} u : A$ and u is SN (i.e. strongly normalizing), then $\langle \Gamma; (\lambda x.s)ut_1 \dots t_n \rangle \in \mathsf{Red}_X$. Since X is a ground type, according to the definition of Red_X we show that:

1. $(\lambda x.s)ut_1 \dots t_n$ is SN, and
2. $\Gamma \vdash_{\text{Church}} (\lambda x^A.s)ut_1 \dots t_n : X$.

The hypothesis $\langle \Gamma; s\{u/x\}t_1 \dots t_n \rangle \in \mathsf{Red}_X$ implies that $s\{u/x\}t_1 \dots t_n$ is SN, by definition of Red_X. Since u is SN by hypothesis too, $(\lambda x^A.s)ut_1 \dots t_n$ is SN by Exercise 6.3 below (adapted to Church-style).

The hypothesis $\langle \Gamma; s\{u/x\}t_1 \dots t_n \rangle \in \mathsf{Red}_X$ implies that $\Gamma \vdash_{\text{Church}} s\{u/x\}t_1 \dots t_n : X$ and hence $\Gamma \vdash_{\text{Church}} s\{u/x\} : A_1 \Rightarrow \dots \Rightarrow A_n \Rightarrow X$ and $\Gamma \vdash_{\text{Church}} t_i : A_i$ for all $1 \leq i \leq n$. By the lemma below, since $\Gamma \vdash_{\text{Church}} u : A$ by hypothesis, $\Gamma, x : A \vdash_{\text{Church}} s : A_1 \Rightarrow \dots \Rightarrow A_n \Rightarrow X$. Thus, the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, x : A \vdash s : A_1 \Rightarrow \dots \Rightarrow A_n \Rightarrow X}\lambda \quad \vdots}{\Gamma \vdash \lambda x^A.s : A \Rightarrow A_1 \Rightarrow \dots \Rightarrow A_n \Rightarrow X \quad \Gamma \vdash u : A}@ \quad \vdots}{\Gamma \vdash (\lambda x^A.s)u : A_1 \Rightarrow \dots \Rightarrow A_n \Rightarrow X \quad \Gamma \vdash t_1 : A_1}@}{\Gamma \vdash (\lambda x^A.s)ut_1 : A_2 \Rightarrow \dots \Rightarrow A_n \Rightarrow X}}{\vdots}}{\Gamma \vdash (\lambda x^A.s)ut_1 \dots t_{n-1} : A_n \Rightarrow X \quad \Gamma \vdash t_n : A_n}@}{\Gamma \vdash (\lambda x^A.s)ut_1 \dots t_n : X}$$

shows that $\Gamma \vdash_{\text{Church}} (\lambda x^A.s)ut_1 \dots t_n : X$.

Lemma. *If* $\Gamma \vdash_{\text{Church}} s\{u/x\} : B$ *and* $\Gamma \vdash_{\text{Church}} u : A$ *then* $\Gamma, x : A \vdash_{\text{Church}} s : B$.

Proof. By induction on s. Cases:

- $s = x$: Then $s\{u/x\} = u$ and $A = B$ by uniqueness of type and derivation in $\mathsf{STLC}_{\text{Church}}$ seen in Lecture 3, p. 19 (because of the hypothesis $\Gamma \vdash_{\text{Church}} s\{u/x\} : B$ and $\Gamma \vdash_{\text{Church}} u : A$), hence the derivation $\dfrac{}{\Gamma, x : A \vdash x : A}\,\mathsf{var}$ shows that $\Gamma, x : A \vdash_{\text{Church}} s : B$.
- $s = y \neq x$: Then $s\{u/x\} = y$ and $\Gamma = \Delta, y : B$ (because of the hypothesis $\Gamma \vdash_{\text{Church}} s\{u/x\} : B$), hence the derivation $\dfrac{}{\Delta, y : B, x : A \vdash y : B}\,\mathsf{var}$ shows that $\Gamma, x : A \vdash_{\text{Curry}} s : B$.
- $s = \lambda y.s_0$: We can suppose without loss of generality that $y \notin \mathsf{dom}(\Gamma) \cup \mathsf{fv}(u) \cup \{x\}$. Then $s\{u/x\} = \lambda y.s_0\{u/x\}$ and $B = B_1 \Rightarrow B_2$ (because of $\Gamma \vdash_{\text{Church}} s\{u/x\} : B$), hence $\Gamma, y : B_1 \vdash_{\text{Church}} s_0\{u/x\} : B_2$. From $\Gamma \vdash_{\text{Church}} u : A$ it follows that $\Gamma, y : B_1 \vdash_{\text{Church}} u : A$. By induction hypothesis, $\Gamma, y : B_1, x : A \vdash_{\text{Church}} s_0 : B_2$.

  Therefore, the derivation $\dfrac{\dfrac{\vdots}{\Gamma, y : B_1, x : A \vdash s_0 : B_2}}{\Gamma, x : A \vdash \lambda y.s_0 : B_1 \Rightarrow B_2}\lambda$ shows that $\Gamma, x : A \vdash_{\text{Church}} s : B$.

- $s = s_1 s_2$: Then $s\{u/x\} = s_1\{u/x\}s_2\{u/x\}$, hence $\Gamma \vdash_{\text{Church}} s_1\{u/x\} : C \Rightarrow B$ and $\Gamma \vdash_{\text{Curry}} s_2\{u/x\} : C$. By induction hypothesis, $\Gamma, x : A \vdash_{\text{Church}} s_1 : C \Rightarrow B$ and $\Gamma, x : A \vdash_{\text{Church}} s_2 : C$. Therefore, the derivation
$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma, x : A \vdash s_1 : C \Rightarrow B & \Gamma, x : A \vdash s_2 : C\end{array}}{\Gamma, x : A \vdash s_1 s_2 : B}\,@$$
shows that $\Gamma, x : A \vdash_{\text{Church}} s : B$. □

Non-assessed exercises from Lecture 3

(https://cl-teaching.uibk.ac.at/26w/lc/slides/03.pdf)

Exercise 1

For each the λ-terms $\lambda x^X.\lambda y^Y.x$ and $\lambda x^{Z \Rightarrow Y \Rightarrow X}.\lambda y^{Z \Rightarrow Y}.\lambda z^Z.xz(yz)$, write the unique derivation in $\mathsf{STLC}_{\text{Church}}$ typing it with some type and the empty environment.

Solution to Exercise 1

$$\dfrac{\dfrac{\dfrac{}{x : X,\, y : Y \vdash x : X}\,\text{var}}{x : X \vdash \lambda y^Y.x : Y \Rightarrow X}\,\lambda}{\vdash \lambda x^X.\lambda y^Y.x : X \Rightarrow Y \Rightarrow X}\,\lambda$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : Z \Rightarrow Y \Rightarrow X}\,\text{var} \quad \dfrac{}{\Gamma \vdash z : Z}\,\text{var}}{\Gamma \vdash xz : Y \Rightarrow X}\,@ \quad \dfrac{\dfrac{}{\Gamma \vdash y : Z \Rightarrow Y}\,\text{var} \quad \dfrac{}{\Gamma \vdash z : Z}\,\text{var}}{\Gamma \vdash yz : Y}\,@}{x : Z \Rightarrow Y \Rightarrow X,\, y : Z \Rightarrow Y,\, z : Z \vdash xz(yz) : X}\,@}{x : Z \Rightarrow Y \Rightarrow X,\, y : Z \Rightarrow Y \vdash \lambda z^Z.xz(yz) : Z \Rightarrow X}\,\lambda}{x : Z \Rightarrow Y \Rightarrow X \vdash \lambda y^{Z \Rightarrow Y}.\lambda z^Z.xz(yz) : (Z \Rightarrow Y) \Rightarrow Z \Rightarrow X}\,\lambda}{\vdash \lambda x^{Z \Rightarrow Y \Rightarrow X}.\lambda y^{Z \Rightarrow Y}.\lambda z^Z.xz(yz) : (Z \Rightarrow Y \Rightarrow X) \Rightarrow (Z \Rightarrow Y) \Rightarrow Z \Rightarrow X}\,\lambda$$

where $\Gamma = x : Z \Rightarrow Y \Rightarrow X,\, y : Z \Rightarrow Y,\, z : Z$.

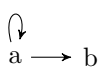
Exercise 2

Define five ARSs $(A, \to)$ such that:

1. in the first, $\to$ is normalizing but not strongly normalizing;
2. in the second, $\to$ is not normalizing;
3. in the third, $\to$ is strongly normalizing;
4. in the fourth, $\to$ is not confluent but every $t \in A$ has at most one normal form;
5. in the fifth, $\to$ is weakly confluent but not confluent (*hint:* it is not SN, by Newman's lemma).

Solution to Exercise 2

1. Let $A = \{a, b\}$ and $\to$ be defined as below.

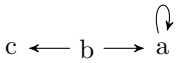


2. Let $A = \{a\}$ and $\to$ be defined as below.



3. Let $A = \{a\}$ and $\to$ be defined as below.

4. Let $A = \{a, b, c\}$ and $\to$ be defined as below.



5. Let $A = \{a, b, c, d\}$ and $\to$ be defined as below.



Exercise 3

In a ARS $(A, \to)$, prove that $t \in A$ is SN (strongly normalizing) iff for every $t' \in A$, if $t \to t'$ then t' is SN.

Solution to Exercise 3

Proof. $\Rightarrow$**:** Let $t \to t'$. If t' is not SN then there is an infinite β-reduction sequence from t'. Appending t to it gives rise to an infinite β-reduction sequence from t, hence t is not SN.

$\Leftarrow$**:** If t is not SN then there is an infinite β-reduction sequence from t. In particular, $t \to t'$ for some $t' \in A$ such that there is an infinite β-reduction sequence from t'. So, there is $t' \in A$ such that $t \to t'$ and t' is not SN. □

Exercise 4

Prove that (in Curry-style λ-terms) if $t \to_\beta t'$ then $t\{s/x\} \to_\beta t'\{s/x\}$ and $s\{t'/x\} \to_\beta s\{t'/x\}$.

Solution to Exercise 4

We prove the two properties in two propositions separately. We first prove the lemma below.

Lemma. *If $y \notin \mathsf{fv}(s) \cup \{x\}$ then $t\{u/y\}\{s/x\} = t\{s/x\}\{u\{s/x\}/y\}$.*

Proof. By induction on t. Cases:

- $t = y$: then $t\{u/y\} = u$ and $t\{s/x\} = y$ (since $y \neq x$), hence $t\{u/y\}\{s/x\} = u\{s/x\} = t\{s/x\}\{u\{s/x\}/y\}$.
- $t = x$: then $t\{u/y\} = x$ (since $y \neq x$) and $t\{s/x\} = s$, hence $t\{u/y\}\{s/x\} = x\{s/x\} = s = t\{s/x\}\{u\{s/x\}/y\}$, where the last equality holds because $y \notin \mathsf{fv}(s)$.
- $t = z \notin \{x, y\}$: then $t\{u/y\} = z = t\{s/x\}$, hence $t\{u/y\}\{s/x\} = z = t\{s/x\}\{u\{s/x\}/y\}$.
- $t = \lambda z.r$ and we can suppose without loss of generality that $z \notin \mathsf{fv}(u) \cup \mathsf{fv}(s) \cup \{x, y\}$: then $t\{u/y\} = \lambda z.r\{u/y\}$ and $t\{s/x\} = \lambda z.r\{s/x\}$. By induction hypothesis, $r\{u/y\}\{s/x\} = r\{s/x\}\{u\{s/x\}/y\}$. Therefore, $t\{u/y\}\{s/x\} = \lambda z.r\{u/y\}\{s/x\} = \lambda z.r\{s/x\}\{u\{s/x\}/y\} = t\{s/x\}\{u\{s/x\}/y\}$.
- $t = t_1 t_2$: then $t\{u/y\} = t_1\{u/y\}t_2\{u/y\}$ and $t\{s/x\} = t_1\{s/x\}t_2\{s/x\}$. By induction hypothesis, $t_i\{u/y\}\{s/x\} = t_i\{s/x\}\{u\{s/x\}/y\}$ for all $1 \leq i \leq 2$. Therefore,

$$t\{u/y\}\{s/x\} = t_1\{u/y\}\{s/x\}t_2\{u/y\}\{s/x\} = t_1\{s/x\}\{u\{s/x\}/y\}t_2\{s/x\}\{u\{s/x\}/y\} = t\{s/x\}\{u\{s/x\}/y\}.$$

□

Proposition. *In Curry-style λ-terms, if $t \to_\beta t'$ then $t\{s/x\} \to_\beta t'\{s/x\}$.*

Proof. By induction on t. Note that t cannot be a variable (which is β-normal) because of the hypothesis $t \to_\beta t'$. Cases:

- $t = \lambda y.u \to_\beta \lambda y.u' = t'$ with $u \to_\beta u'$. We can suppose without loss of generality that $y \notin \mathsf{fv}(s) \cup \{x\}$. By induction hypothesis, $u\{s/x\} \to_\beta u'\{s/x\}$. Hence, $t\{s/x\} = \lambda y.(u\{s/x\}) \to_\beta \lambda y.(u'\{s/x\}) = t'\{s/x\}$.
- $t = t_1 t_2 \to_\beta t_1' t_2 = t'$ with $t_1 \to_\beta t_1'$. By induction hypothesis, $t_1\{s/x\} \to_\beta t_1'\{s/x\}$. Hence, $t\{s/x\} = t_1\{s/x\}t_2\{s/x\} \to_\beta t_1'\{s/x\}t_2\{s/x\} = t'\{s/x\}$.

- $t' = t_1 t_2 \to_\beta t_1 t_2' = t'$ with $t_2 \to_\beta t_2'$. Analogous to the previous case.
- $t = (\lambda y.t_1)t_2 \to_\beta t_1\{t_2/y\} = t'$. We can suppose without loss of generality that $y \notin \mathsf{fv}(s) \cup \{x\}$. Thus, $t\{s/x\} = (\lambda y.t_1\{s/x\})t_2\{s/x\} \to_\beta t_1\{s/x\}\{t_2\{s/x\}/y\} = t_1\{t_2/y\}\{s/x\} = t'\{s/x\}$ where the second to last equality holds by the lemma above. □

Proposition. *In Curry-style λ-terms, if $t \to_\beta t'$ then $s\{t/x\} \to_\beta^* s\{t'/x\}$.*

Proof. By induction on s. Cases:

- $s = x$. Then, $s\{t/x\} = t \to_\beta t' = s\{t'/x\}$ by hypothesis, hence $s\{t/x\} \to_\beta^* s\{t'/x\}$.
- $s = y \neq x$. Then, $s\{t/x\} = y = s\{t'/x\}$, hence $s\{t/x\} \to_\beta^* s\{t'/x\}$.
- $s = \lambda y.u$. We can suppose without loss of generality that $y \notin \mathsf{fv}(t) \cup \{x\}$. By induction hypothesis, $u\{t/x\} \to_\beta^* u'\{t/x\}$. Hence, $s\{t/x\} = \lambda y.(u\{t/x\}) \to_\beta^* \lambda y.(s\{t'/x\}) = s\{t'/x\}$.
- $s = s_1 s_2$. By induction hypothesis, $s_1\{t/x\} \to_\beta^* s_1\{t'/x\}$ and $s_2\{t/x\} \to_\beta^* s_2\{t'/x\}$. Therefore, $s\{t'/x\} = s_1\{t/x\}s_2\{t/x\} \to_\beta^* s_1\{t'/x\}s_2\{t/x\} \to_\beta^* s_1\{t'/x\}s_2\{t/'\}x = s\{t'/x\}$. □

Exercise 5

Prove that if a λ-term t (in Curry-style) is SN, then so is every subterm of t.

Solution to Exercise 5

Let us recall that the *subterms* of a λ-term t are the elements of the set $\mathsf{subt}(t)$ defined below by induction on t:

$$\mathsf{subt}(x) = \{x\} \qquad \mathsf{subt}(\lambda x.t) = \{\lambda x.t\} \cup \mathsf{subt}(t) \qquad \mathsf{subt}(st) = \{st\} \cup \mathsf{subt}(s) \cup \mathsf{subt}(t)$$

We now prove that if a λ-term t (in Curry-style) is SN, then so is every subterm of t. We proceed by induction on t. Let $s \in \mathsf{subt}(t)$. Cases:

- If $s = t$ then s is SN by hypothesis.
- If $s \neq t = \lambda x.t'$ then $s \in \mathsf{subt}(t')$. Since t is SN, then so is t' (indeed any infinite β-reduction sequence from t' lifts to an infinite β-reduction sequence from t). By induction hypothesis applied to t', s is SN.
- If $s \neq t = t_1 t_2$ then $s \in \mathsf{subt}(t_1) \cup \mathsf{subt}(t_2)$. Since t is SN, then so are t_1 and t_2 (indeed any infinite β-reduction sequence from t_1 or t_2 lifts to an infinite β-reduction sequence from t). By induction hypothesis applied to t_1 or t_2—depending on whether $s \in \mathsf{subt}(t_1)$ or $s \in \mathsf{subt}(t_2)$—, s is SN.

This complete the proof.

Remark. *Note that the case $t = s$ includes the case where t is a variable, since the only subterm of a variable is the variable itself.*

Exercise 6

Prove the following facts for λ-terms (in Curry-style):

1. for all $n \in \mathbb{N}$, if $t_1, \ldots, t_n$ are SN, then so is $xt_1 \ldots t_n$;
2. if s is SN, then so is $\lambda x.s$;
3. for all $n \in \mathbb{N}$, if t and $s\{t/x\}t_1 \ldots t_n$ are SN, then so is $(\lambda x.s)tt_1 \ldots t_n$;
4. for all $n \in \mathbb{N}$, if $x \in \mathsf{fv}(s)$ and $s\{t/x\}t_1 \ldots t_n$ is SN then so is $(\lambda x.s)tt_1 \ldots t_n$.

Solution to Exercise 6

1. If $xt_1 \dots t_n$ is not SN, then there is an infinite β-reduction sequence ρ from $xt_1 \dots t_n$. As a β-redex in $xt_1 \dots t_n$ is a β-redex in t_i for some $1 \leq i \leq n$, then ρ induces an infinite β-reduction sequence from t_i for some $1 \leq i \leq n$. Hence, t_i is not SN for some $1 \leq i \leq n$.

2. If s is not SN, then there is an infinite β-reduction sequence from s, which lifts to an infinite β-reduction sequence from $t = \lambda x.s$. Hence, t is not SN.

3. If $u = (\lambda x.s)tt_1 \dots t_n$ is not SN, then there is an infinite β-reduction sequence from u. There are the following cases, since any β-redex in u is either $(\lambda x.s)t$ or a β-redex in s or t or t_i for some $1 \leq i \leq n$:

   - either $u \to_\beta (\lambda x.s)t't_1 \dots t_n$ where $t \to_\beta t'$ and there is an infinite β-reduction sequence from t';
   - or $u \to_\beta (\lambda x.s)tt_1 \dots t'_i \dots t_n$ where $t_i \to_\beta t'_i$ and there is an infinite β-reduction sequence from t'_i, for some $1 \leq i \leq n$;
   - or $u \to_\beta (\lambda x.s')tt_1 \dots t_n$ where $s \to_\beta s'$ and there is an infinite β-reduction sequence from s';
   - or $u \to_\beta s\{t/x\}t_1 \dots t_n$ and there is an infinite β-reduction sequence from $s\{t/x\}t_1 \dots t_n$.

   In any case, there is an infinite β-reduction sequence from t or from $s\{t/x\}t_1 \dots t_n$. Therefore, t is not SN or $s\{t/x\}t_1 \dots t_n$ is not SN.

4. If $u = (\lambda x.s)tt_1 \dots t_n$ is not SN, then there is an infinite β-reduction sequence from u. Since any β-redex in u is either $(\lambda x.s)t$ or a β-redex in s or t or t_i for some $1 \leq i \leq n$, such an infinite β-reduction sequence is from $s\{t/x\}$ or from s or from t or from t_i for some $1 \leq i \leq n$. Since $x \in \mathsf{fv}(s)$, then $t \in \mathsf{subt}(s\{t/x\})$, hence an infinite β-reduction sequence from t lifts to an infinite β-reduction sequence from $s\{t/x\}$. In any case, there is an infinite β-reduction sequence from $s\{t/x\}t_1 \dots t_n$, hence $s\{t/x\}t_1 \dots t_n$ is not SN.

Exercise 7

Prove that if t can be built up from the rules below applied $n \in \mathbb{N}$ times, then t is SN and the maximal length $\mathsf{sn}(t)$ of the β-reduction sequences from t (to its β-normal form) is $\leq n$.

$$\frac{(t_i)_{1 \leq i \leq k}}{xt_1 \dots t_k}\,@_k \qquad \frac{t}{\lambda x.t}\,\lambda \qquad \frac{u \qquad s\{u/x\}t_1 \dots t_k}{(\lambda x.s)ut_1 \dots t_k}\,\beta_k \qquad \text{(for any } k \in \mathbb{N} \text{ in the rules } @_k \text{ and } \beta_k\text{)}$$

Deduce that the set of SN λ-terms is the least set closed under those 3 rules.

Solution to Exercise 7

For every derivation D built up from the rules above, let $|D|$ denote the number of rules in D. We say that the derivation D *derives* t if t is the conclusion of D. To solve the exercise, we first prove the lemma below.

Lemma. *if $t \to_\beta t'$ and D derives t, then there is a derivation D' deriving t' such that $|D| > |D'|$.*

Proof. By induction on the derivation D deriving t. Let us consider its last rule r. Cases:

- $\mathsf{r} = @_k$ for some $k \in \mathbb{N}$: then D derives $t = xt_1 \dots t_k$ and D_i derives t_i for all $1 \leq i \leq k$, with $|D| = 1 + \sum_{i=1}^{k} |D_i|$. As $t \to_\beta t'$, necessarily $t' = xt_1 \dots t'_j \dots t_k$ with $t_j \to_\beta t'_j$ for some $1 \leq j \leq k$. By induction hypothesis, there is D'_j deriving t'_i and $|D_j| > |D'_j|$. By applying the rule $@_k$ on derivations $D_1, \dots, D'_j, \dots, D_k$, we obtain a derivation D' deriving t' with $|D'| = 1 + |D'_j| + \sum_{\substack{i=1 \\ i \neq j}}^{k} |D_i| < 1 + \sum_{i=1}^{k} |D_i| = |D|$.

- $\mathsf{r} = \lambda$: then D derives $t = \lambda x.t_0$ and D_0 derives t_0 with $|D| = 1 + |D_0|$. As $t \to_\beta t'$, necessarily $t' = \lambda x.t_0$ with $t_0 \to_\beta t'_0$. By induction hypothesis, there is D'_0 deriving t'_0 and $|D_0| > |D'_0|$. By applying the rule λ on the derivation D'_0, we obtain a derivation D' deriving t' with $|D'| = 1 + |D'_0| < 1 + |D_0| = |D|$.

- $\mathsf{r} = \beta_k$ for some $k \in \mathbb{N}$: then D derives $t = (\lambda x.s)ut_1 \dots t_k$, and D_1 and D_2 derive u and $s\{u/x\}t_1 \dots t_k$, respectively, with $|D| = 1 + |D_1| + |D_2|$. There are several subcases for $t \to_\beta t'$:

  - $t' = (\lambda x.s)ut_1 \dots t'_j \dots t_k$ with $t_j \to_\beta t'_j$ for some $1 \leq j \leq k$. As $s\{u/x\}t_1 \dots t_k \to_\beta s\{u/x\}t_1 \dots t'_j \dots t_k$, by induction hypothesis there is D'_2 deriving $s\{u/x\}t_1 \dots t'_j \dots t_k$ and $|D_2| > |D'_2|$. By applying the rule β_k to the derivations D_1 and D'_2, we obtain a derivation D' deriving t' with $|D'| = 1 + |D_1| + |D'_2| < 1 + |D_1| + |D_2| = |D|$.

– $t' = (\lambda x.s)u't_1 \dots t_k$ with $u \to_\beta u'$. By induction hypothesis, there is D_1' deriving u' and $|D_1| > |D_1'|$. By Exercise 4, $s\{u/x\}t_1 \dots t_k \to_\beta^* s\{u'/x\}t_1 \dots t_k$. By applying the induction hypothesis $n \in \mathbb{N}$ times, where n is the number of β-steps from $s\{u/x\}t_1 \dots t_k$ to $s\{u'/x\}t_1 \dots t_k$, there is D_2' deriving $s\{u'/x\}t_1 \dots t_k$ and $|D_2| \geq |D_2'|$. By applying the rule β_k to the derivations D_1 and D_2', we obtain a derivation D' deriving t' with $|D'| = 1 + |D_1'| + |D_2'| < 1 + |D_1| + |D_2| = |D|$.

– $t' = (\lambda x.s')ut_1 \dots t_k$ with $s \to_\beta s'$. By Exercise 4, $s\{u/x\}t_1 \dots t_k \to_\beta s'\{u/x\}t_1 \dots t_k$. By induction hypothesis, there is D_2' deriving $s\{u'/x\}t_1 \dots t_k$ and $|D_2| > |D_2'|$. By applying the rule β_k to derivations D_1 and D_2', we obtain a derivation D' deriving t' with $|D'| = 1 + |D_1| + |D_2'| < 1 + |D_1| + |D_2| = |D|$.

– $t' = s\{u/x\}t_1 \dots t_k$. Then D_2 derives t' and $|D_2| < 1 + |D_1| + |D_2| = |D|$. □

Proposition. *If a λ-term t can be constructed by applying $n \in \mathbb{N}$ times the rules above, then t is SN and $\mathsf{sn}(t) \leq n$.*

Proof. By induction on n. If t is β-normal, then t is SN and $\mathsf{sn}(t) = 0 \leq n$. Otherwise $t \to_\beta t'$; by the lemma above, there is a derivation D' deriving t' with $|D'| < n$; by induction hypothesis, t' is SN and $\mathsf{sn}(t') \leq |D'|$. Since t' is SN and $\mathsf{sn}(t') \leq |D'|$ for every $t' \in t_{\to_\beta} = \{t' \mid t \to_\beta t'\}$, by Exercise 3 t is SN, and $\mathsf{sn}(t) = 1 + \max_{t' \in t_{\to_\beta}} \mathsf{sn}(t') < 1 + n$, therefore $\mathsf{sn}(t) \leq n$. □

Exercise 8

Prove that the SN λ-terms (in Curry-style) are exactly those built up from the rules $@_k, \lambda, \beta_k$ above.

Solution to Exercise 8

By Exercise 6.1–3 (or alternatively, Exercise 7), all the λ-terms built up from the rules $@_k, \lambda, \beta_k$ above are in Curry-style and SN.

Let t be a SN λ-term in Curry-style. Cases:

- $t = xt_1 \dots t_k$ for some $1 \leq i \leq k$: necessarily t_i is SN and in Curry-style for all $1 \leq i \leq k$. By induction hypothesis, t_i is built up from the rules $@_n, \lambda, \beta_n$, for all $1 \leq i \leq k$. By applying the rule $@_k$ to $t_1, \dots, t_k$, the λ-term t is built up from the rules $@_n, \lambda, \beta_n$.

- $t = (\lambda x.s)ut_1 \dots t_k$ for some $1 \leq i \leq k$: necessarily $s, u, t_1, \dots, t_k$ are SN and in Curry-style. Since $t \to_\beta s\{u/x\}t_1 \dots t_k$ and t is SN, $s\{u/x\}t_1 \dots t_k$ is SN (an in Curry-style). By induction hypothesis, u and $s\{u/x\}t_1 \dots t_k$ are built up from the rules $@_n, \lambda, \beta_n$. By applying the rule β_k to u and $s\{u/x\}t_1 \dots t_k$, the λ-term t is built up from the rules $@_n, \lambda, \beta_n$.

- $t = \lambda x.t_0$: necessarily t_0 is SN and in Curry-style. By induction hypothesis, t_0 is built up from the rules $@_n, \lambda, \beta_n$. By applying the rule λ to t_0, the λ-term t is built up from the rules $@_n, \lambda, \beta_n$.