

The λ -calculus: typed and untyped

<https://cl-teaching.uibk.ac.at/26w/lc/>

Solutions to selected exercises from Lecture 2

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Assessed exercises from Lecture 2

(<https://cl-teaching.uibk.ac.at/26w/lc/exercises/02.pdf>)

Exercise 1 (3 points)

Find the simply typed λ -terms, both in Curry-style *and* in Church-style, associated with the derivation $\mathcal{D} \triangleright_{\text{ND}} ((X \Rightarrow Y) \Rightarrow X)^x \vdash (X \Rightarrow Y) \Rightarrow Y$ found for the assessed Exercise 1 from Lecture 1.

Solution to Exercise 1

It is too early to reveal a model solution to this exercise.

Exercise 2 (7 points)

Consider the proof system $\text{ND}^\wedge$ for minimal logic extended with conjunction, as defined in the assessed Exercises 2–3 from Lecture 1, and the term language of the λ -calculus extended with pairs and projections defined below.

$$s, t ::= x \mid \lambda x. t \mid st \mid \langle s, t \rangle \mid \pi_1(t) \mid \pi_2(t) \quad (1)$$

1. (3 points) Define how to decorate formulas in the inference rules of $\text{ND}^\wedge$ with the terms of the grammar (1).
2. (4 points) Extend β -reduction to terms of the grammar (1), so as that the decoration defined in Item 1 commutes with cut-elimination rules defined in the assessed Exercise 3 from Lecture 1.

Solution to Exercise 2

It is too early to reveal a model solution to this exercise.

Non-assessed exercises from Lecture 2

(<https://cl-teaching.uibk.ac.at/26w/lc/slides/02.pdf>)

Exercise 1

Find the simply typed λ -terms (in Curry-style and Church-style) associated with the derivations in ND found for the facts below (see Exercise 1 from Lecture 1).

1. $\vdash X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)$.
2. $((X \Rightarrow Y) \Rightarrow X \Rightarrow Z)^z \vdash Y \Rightarrow X \Rightarrow Z$.
3. $((X \Rightarrow Y) \Rightarrow X)^z \vdash Y \Rightarrow X$.
4. $(X \Rightarrow Y \Rightarrow Z)^z \vdash Y \Rightarrow X \Rightarrow Z$.
5. $(X \Rightarrow Y \Rightarrow Z)^z, (X \Rightarrow Y)^y \vdash X \Rightarrow Z$.
6. $((X \Rightarrow X) \Rightarrow Y)^y \vdash (Y \Rightarrow Z) \Rightarrow Z$.

Solution to Exercise 1

1. In Curry-style and Church-style for λ -terms, and ND for derivations:

$$\frac{\frac{\frac{[y : (X \Rightarrow Y)^y] \quad [x : X^x]}{yx : Y} \Rightarrow_e}{\lambda y.yx : (X \Rightarrow Y) \Rightarrow Y} \Rightarrow_i^y}{\lambda x.\lambda y.yx : X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)} \Rightarrow_i^x \quad \frac{\frac{\frac{[y : (X \Rightarrow Y)^y] \quad [x : X^x]}{yx : Y} \Rightarrow_e}{\lambda y^{X \Rightarrow Y}.yx : (X \Rightarrow Y) \Rightarrow Y} \Rightarrow_i^y}{\lambda x^X.\lambda y^{X \Rightarrow Y}.yx : X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)} \Rightarrow_i^x$$

2. In Curry-style and Church-style for λ -terms, and ND for derivations:

$$\frac{\frac{z : ((X \Rightarrow Y) \Rightarrow X \Rightarrow Z)^z \quad \frac{[y : Y^y]}{\lambda x.y : X \Rightarrow Y} \Rightarrow_i^x}{z(\lambda x.y) : X \Rightarrow Z} \Rightarrow_e}{\lambda y.z(\lambda x.y) : Y \Rightarrow X \Rightarrow Z} \Rightarrow_i^y \quad \frac{\frac{z : ((X \Rightarrow Y) \Rightarrow X \Rightarrow Z)^z \quad \frac{[y : Y^y]}{\lambda x^X.y : X \Rightarrow Y} \Rightarrow_i^x}{z(\lambda x^X.y) : X \Rightarrow Z} \Rightarrow_e}{\lambda y^Y.z(\lambda x^X.y) : Y \Rightarrow X \Rightarrow Z} \Rightarrow_i^y$$

3. In Curry-style and Church-style for λ -terms, and ND for derivations:

$$\frac{\frac{z : ((X \Rightarrow Y) \Rightarrow X)^z \quad \frac{[y : Y^y]}{\lambda x.y : X \Rightarrow Y} \Rightarrow_i^x}{z(\lambda x.y) : X} \Rightarrow_e}{\lambda y.z(\lambda x.y) : Y \Rightarrow X} \Rightarrow_i^y \quad \frac{\frac{z : ((X \Rightarrow Y) \Rightarrow X)^z \quad \frac{[y : Y^y]}{\lambda x^X.y : X \Rightarrow Y} \Rightarrow_i^x}{z(\lambda x^X.y) : X} \Rightarrow_e}{\lambda y^Y.z(\lambda x^X.y) : Y \Rightarrow X} \Rightarrow_i^y$$

4. In Curry-style and Church-style for λ -terms, and ND for derivations:

$$\frac{\frac{\frac{z : (X \Rightarrow Y \Rightarrow Z)^z \quad [x : X^x]}{zx : Y \Rightarrow Z} \Rightarrow_e \quad [y : Y^y]}{zxy : Z} \Rightarrow_e}{\frac{\frac{zxy : Z}{\lambda x.zxy : X \Rightarrow Z} \Rightarrow_i^x}{\lambda y.\lambda x.zxy : Y \Rightarrow X \Rightarrow Z} \Rightarrow_i^y} \quad \frac{\frac{\frac{z : (X \Rightarrow Y \Rightarrow Z)^z \quad [x : X^x]}{zx : Y \Rightarrow Z} \Rightarrow_e \quad [y : Y^y]}{zxy : Z} \Rightarrow_e}{\frac{\frac{zxy : Z}{\lambda x^X.zxy : X \Rightarrow Z} \Rightarrow_i^x}{\lambda y^Y.\lambda x^X.zxy : Y \Rightarrow X \Rightarrow Z} \Rightarrow_i^y}$$

5. In Curry-style and Church-style for λ -terms, and ND for derivations:

$$\frac{\frac{z : (X \Rightarrow Y \Rightarrow Z)^z \quad [x : X^x]}{zx : Y \Rightarrow Z} \Rightarrow_e \quad \frac{y : (X \Rightarrow Y)^y \quad [x : X^x]}{yx : Y} \Rightarrow_e}{\frac{zxy : Z}{\lambda x.zxy : X \Rightarrow Z} \Rightarrow_i^x} \quad \frac{\frac{z : (X \Rightarrow Y \Rightarrow Z)^z \quad [x : X^x]}{zx : Y \Rightarrow Z} \Rightarrow_e \quad \frac{y : (X \Rightarrow Y)^y \quad [x : X^x]}{yx : Y} \Rightarrow_e}{\frac{zxy : Z}{\lambda x^X.zxy : X \Rightarrow Z} \Rightarrow_i^x} \Rightarrow_e$$

6. In Curry-style and Church-style for λ -terms, and ND for derivations:

$$\frac{\frac{[z : (Y \Rightarrow Z)^z] \quad \frac{y : ((X \Rightarrow X) \Rightarrow Y)^y \quad \frac{[x : X^x]}{\lambda x.x : X \Rightarrow X} \Rightarrow_i^x}{y \lambda x.x : Y} \Rightarrow_e}{z(y \lambda x.x) : Z} \Rightarrow_e}{\lambda z.z(y \lambda x.x) : (Y \Rightarrow Z) \Rightarrow Z} \Rightarrow_i^z \quad \frac{\frac{[z : (Y \Rightarrow Z)^z] \quad \frac{y : ((X \Rightarrow X) \Rightarrow Y)^y \quad \frac{[x : X^x]}{\lambda x^X.x : X \Rightarrow X} \Rightarrow_i^x}{y \lambda x^X.x : Y} \Rightarrow_e}{z(y \lambda x^X.x) : Z} \Rightarrow_e}{\lambda z^{Y \Rightarrow Z}.z(y \lambda x^X.x) : (Y \Rightarrow Z) \Rightarrow Z} \Rightarrow_i^z$$

Exercise 2

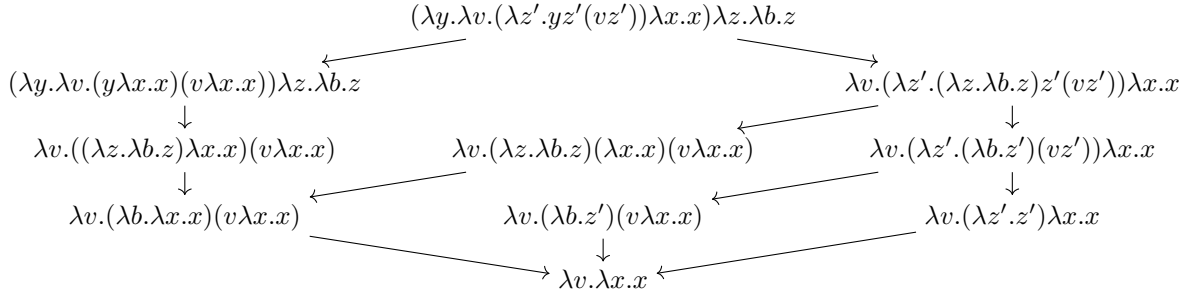
Perform all possible β -reduction steps from the λ -term decorating the derivation $\mathcal{D}$ in ND on p. 31 of Lecture 1, until you get a β -normal form. Is it always the same? Compare it with the normal derivation obtained by cut-elimination steps from $\mathcal{D}$.

Solution to Exercise 2

The derivation on p. 31 of Lecture 1 slides is $\mathcal{D}$ below, decorated with λ -terms in Curry-style.

$$\begin{array}{c}
 \frac{[y : ((X \Rightarrow X) \Rightarrow B \Rightarrow X \Rightarrow X)^y] \ [z' : (X \Rightarrow X)^{z'}]}{yz' : B \Rightarrow (X \Rightarrow X)} \Rightarrow_e \quad \frac{[v : ((X \Rightarrow X) \Rightarrow B)^v] \ [z' : (X \Rightarrow X)^{z'}]}{vz' : B} \Rightarrow_e \\
 \frac{yz' : B \Rightarrow (X \Rightarrow X)}{yz'(vz') : X \Rightarrow X} \Rightarrow_i^{z'} \quad \frac{[x : X^x]}{\lambda x.x : X \Rightarrow X} \Rightarrow_i^x \\
 \frac{\lambda z'.yz'(vz') : (X \Rightarrow X) \Rightarrow (X \Rightarrow X)}{(\lambda z'.yz'(vz'))\lambda x.x : X \Rightarrow X} \Rightarrow_e \\
 \frac{\lambda v.(\lambda z'.yz'(vz'))\lambda x.x : ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\lambda y.\lambda v.(\lambda z'.yz'(vz'))\lambda x.x : ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^y \\
 \frac{[z : (X \Rightarrow X)^z]}{\lambda b.z : B \Rightarrow X \Rightarrow X} \Rightarrow_i^b \\
 \frac{\lambda z.\lambda b.z : (X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}{\lambda z.\lambda b.z : (X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)} \Rightarrow_e \\
 \frac{\lambda y.\lambda v.(\lambda z'.yz'(vz'))\lambda x.x : ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X) \quad \lambda z.\lambda b.z : (X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}{\lambda y.\lambda v.(\lambda z'.yz'(vz'))\lambda x.x : ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_e
 \end{array}$$

Thus, the λ -term decorating $\mathcal{D}$ is $t = (\lambda y.\lambda v.(\lambda z'.yz'(vz'))\lambda x.x)\lambda z.\lambda b.z$. All possible β -reduction steps from t are the following:



In any case, every β -reduction sequence eventually reaches the same β -normal term $\lambda v.\lambda x.x$. Note that $\lambda v.\lambda x.x$ is the decoration of the derivation $\mathcal{D}_0$ below, which is the derivation without redexes to which $\mathcal{D}$ eventually reduces via cut-elimination steps (see Exercise 3 from day 1).

$$\frac{[x : X^x]}{\lambda x.x : X \Rightarrow X} \Rightarrow_i^x \\
 \frac{\lambda x.x : X \Rightarrow X}{\lambda v.\lambda x.x : ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^v$$

Exercise 3

Prove rigorously the following facts ($f^n x = \overbrace{f(\dots(f x)\dots)}^{n \text{ times } f}$) for any $n \in \mathbb{N}$:

1. $\lambda x.xx$ is untypable in Curry-style, $\lambda x^A.xx$ is untypable in Church-style for any type A ;
2. in Church-style, $\lambda f^Y.\lambda x^X.f^n x$ is not typable for any $n > 0$ but $\lambda f^Y.\lambda x^X.x$ is typable;
3. $\lambda f.\lambda x.f^n x$ is typable in Curry-style, for all $n \in \mathbb{N}$.

Solution to Exercise 3

1. *Curry-style:* Suppose by absurd that $\lambda x.xx$ is typable in the simply typed λ -calculus in Curry-style. Then there would be a derivation $\mathcal{D}$ of $\lambda x.xx$. Its last rule is necessarily λ (because the term in the derivation is an abstraction), and its second to last rule is necessarily $@$ (because the body of the abstraction in the derivation is an application), and its leaves are necessarily var rules (because the proper subterms of the application are variables), hence $\mathcal{D}$ has the form below, for some types A, B, C .

$$\frac{\frac{x : A \vdash x : C \Rightarrow B}{x : A \vdash x : C} \text{var} \quad \frac{x : A \vdash x : C}{x : A \vdash x : C} \text{var}}{\frac{x : A \vdash xx : B}{\vdash \lambda x.xx : A \Rightarrow B} \text{@}} \lambda$$

To make $\mathcal{D}$ a valid derivation, the two instances of the rule var must be correct, thus $A = C \Rightarrow B$ and $A = C$ must hold, which implies that $C = C \Rightarrow B$, but this is impossible for any type B, C .

Church-style: Suppose by absurd that $\lambda x^A.xx$ is typable in the simply typed λ -calculus in Church-style. Then there would be a derivation $\mathcal{D}$ of $\lambda x^A.xx$. Its last rule is necessarily λ abstracting a variable of type A (because the term in the derivation is an abstraction of type A), and its second to last rule is necessarily $@$

(because the body of the abstraction is an application), and its leaves are necessarily **var** rules (because the proper subterms of the application are variables), hence $\mathcal{D}$ has the form below, for some types B, C .

$$\frac{\frac{\frac{\overline{x : A \vdash x : C \Rightarrow B}^{\text{var}} \quad \overline{x : A \vdash x : C}^{\text{var}}}{x : A \vdash xx : B}^{\text{@}}}{\vdash \lambda x^A.xx : A \Rightarrow B}^{\lambda}$$

To make $\mathcal{D}$ a valid derivation, the two instances of the rule **var** must be correct, thus $A = C \Rightarrow B$ and $A = C$ must hold, which implies that $C = C \Rightarrow B$, but this is impossible for any type B, C .

2. The term $\lambda f^Y.\lambda x^X.x$ is typable in Church-style, as shown by the derivation below.

$$\frac{\frac{\frac{\overline{f : Y, x : X \vdash x : X}^{\text{var}}}{f : Y \vdash \lambda x^X.x : X \Rightarrow X}^{\lambda}}{\vdash \lambda f^Y.\lambda x^X.x : Y \Rightarrow X \Rightarrow X}^{\lambda}$$

We prove by contradiction that $\lambda f^Y.\lambda x^X.f^n x$ is not typable in Church-style for any $n \in \mathbb{N}^+$. Since $n \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$, then $f^n x = f(f^{n-1}x)$ where $n-1 \in \mathbb{N}$. Suppose by absurd that $\lambda f^Y.\lambda x^X.f^n x$ is typable in the simply typed λ -calculus in Church-style. Then there would be a derivation $\mathcal{D}$ of $\lambda f^Y.\lambda x^X.f^n x$. Its two last rules are necessarily λ (because the term in the derivation is a double abstraction), and its third to last rule is necessarily $@$ (because the body of the double abstraction is the application $f(f^{n-1}x)$), and the left premise of the $@$ rule is necessarily a **var** rule (because the left subterm of the application is a variable), hence $\mathcal{D}$ has the form below, for some types A, B .

$$\frac{\frac{\frac{\overline{f : Y, x : X \vdash f : B \Rightarrow A}^{\text{var}} \quad \frac{\vdots}{f : Y, x : X \vdash f^{n-1}x : B}}{f : Y, x : X \vdash f^n x : A}^{\text{@}}}{\frac{f : Y \vdash \lambda x^X.x : X \Rightarrow A}^{\lambda}}^{\lambda}$$

To make $\mathcal{D}$ a valid derivation, the left instance of the rule **var** must be correct, thus $Y = B \Rightarrow A$ must hold for some types A, B , but this is impossible because Y is a ground type.

3. We first prove the following.

Fact. For all $n \in \mathbb{N}$, there is a derivation of $f : X \Rightarrow X, x : X \vdash f^n x : X$ (in Curry-style and Church-style).

Proof. By induction on $n \in \mathbb{N}$. Cases:

- (a) $n = 0$: then, $f^0 x = x$ and hence the derivation below concludes.

$$\overline{f : X \Rightarrow X, x : X \vdash x : X}^{\text{var}}$$

- (b) $n > 0$: then $f^n x = f(f^{n-1}x)$ and by induction hypothesis there is a derivation $\mathcal{D}$ of $f : X \Rightarrow X, x : X \vdash f^{n-1}x : X$. The derivation below concludes.

$$\frac{\frac{\overline{f : X \Rightarrow X, x : X \vdash f : X \Rightarrow X}^{\text{var}} \quad \frac{\vdots \mathcal{D}}{f : X \Rightarrow X, x : X \vdash f^{n-1}x : X}}{f : X \Rightarrow X, x : X \vdash f^n x : X}^{\text{@}}$$

□

We can now show that, for all $n \in \mathbb{N}$, the term $\lambda f.\lambda x.f^n x$ is typable in Curry-style. Indeed, by the fact above, there is a derivation $\mathcal{D}$ of $f : X \Rightarrow X, x : X \vdash f^n x : X$ for all $n \in \mathbb{N}$. The derivation below concludes:

$$\frac{\frac{\frac{\vdots \mathcal{D}}{f : X \Rightarrow X, x : X \vdash f^n x : X}}{f : X \Rightarrow X \vdash \lambda x.f^n x : X \Rightarrow X}^{\lambda}}{\vdash \lambda f.\lambda x.f^n x : (X \Rightarrow X) \Rightarrow X \Rightarrow X}^{\lambda}$$

Exercise 4

Prove that if t is typable in Church or Curry style, then so is every subterm of t .

Solution to Exercise 4

Let us recall that the *subterms* of a λ -term t are the elements of the set $\text{subt}(t)$ defined below by induction on t :

$$\text{subt}(x) = \{x\} \quad \text{subt}(\lambda x.t) = \{\lambda x.t\} \cup \text{subt}(t) \quad \text{subt}(st) = \{st\} \cup \text{subt}(s) \cup \text{subt}(t)$$

We now prove that if a term t is typable in STLC in Curry style, then so is every subterm of t . The proof for t typable in STLC in Church style is analogous. We proceed by induction on t . Let $s \in \text{subt}(t)$. Cases:

- If $s = t$ then s is typable by hypothesis.
- If $s \neq t = \lambda x.t'$ then $s \in \text{subt}(t')$. Since t is typable in STLC in Curry style, then there is a derivation $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash \lambda x.t' : A$ for some environment Γ and type A . Necessarily, $\mathcal{D}$ has the form below, with $A = B \Rightarrow C$ for some types B, C :

$$\mathcal{D} = \frac{\begin{array}{c} \vdots \\ \mathcal{D}' \end{array}}{\Gamma, x : B \vdash t' : C} \Rightarrow_i \Gamma \vdash \lambda x.t' : B \Rightarrow C$$

Hence, t' is typable. By induction hypothesis applied to t' , s is typable.

- If $s \neq t = t_1 t_2$ then $s \in \text{subt}(t_1) \cup \text{subt}(t_2)$. Since t is typable in STLC in Curry style, then there is a derivation $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash t_1 t_2 : A$ for some environment Γ and type A . Necessarily, $\mathcal{D}$ has the form below:

$$\mathcal{D} = \frac{\begin{array}{c} \vdots \\ \mathcal{D}_1 \end{array} \quad \begin{array}{c} \vdots \\ \mathcal{D}_2 \end{array}}{\Gamma \vdash t_1 : B \Rightarrow A \quad \Gamma \vdash t_2 : B} \Rightarrow_e \Gamma \vdash t_1 t_2 : A$$

Hence, t_1 and t_2 are typable. By induction hypothesis applied to t_1 or t_2 (depending on whether $s \in \text{subt}(t_1)$ or $s \in \text{subt}(t_2)$), s is typable.

This complete the proof.

Remark. Note that the case $t = s$ includes the case where t is a variable, since the only subterm of a variable is the variable itself.

Exercise 5

Prove rigorously the propositions on pp. 21 and 25, the lemma and theorems on p. 26 of the Lecture 2 slides.

Solution to Exercise 5

We first define some notions that will be used in the proofs. The *size* of a derivation in STLC is its number of typing rule instances. A derivation $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ is *minimal* if there is no derivation $\mathcal{D}' \triangleright_{\text{STLC}} \Gamma \vdash t : A$ whose size is less than the size of $\mathcal{D}$.

Proposition.

1. (Weakening) If $\Gamma \vdash_{\text{STLC}} t : A$, then $\Gamma, x : B \vdash_{\text{STLC}} t : A$, for any type B and variable $x \notin \text{dom}(\Gamma)$.
2. (Free variables) If $\Gamma \vdash_{\text{STLC}} t : A$, then $\text{fv}(t) \subseteq \text{dom}(\Gamma)$ and $\Gamma \upharpoonright_{\text{fv}(t)} \vdash_{\text{STLC}} t : A$.

Proof. We prove both statement for STLC in Curry style. The proof for Church style is analogous.

1. Let B be a type and $x \notin \text{dom}(\Gamma)$. We prove the statement by induction on the size of a minimal derivation $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash t : A$. Let us consider the last rule r of $\mathcal{D}$. Cases:

- If $r = \text{var}$ then $\mathcal{D} = \overline{\Gamma', y : A \vdash y : A}^{\text{var}}$ with $\Gamma = \Gamma', y : A$ and $t = y$. Since $x \notin \text{dom}(\Gamma)$, the derivation $\mathcal{D}' = \overline{\Gamma', x : B, y : A \vdash y : A}^{\text{var}}$ is such that $\mathcal{D}' \triangleright_{\text{Curry}} \Gamma, x : B \vdash t : A$.

- If $r = \lambda$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_0}{\Gamma, y : C \vdash s : D}}{\Gamma \vdash \lambda y.s : C \Rightarrow D} \lambda$$

with $A = C \Rightarrow D$ and $t = \lambda y.s$; we can suppose without loss of generality that $y \notin \text{dom}(\Gamma) \cup \{x\}$. By induction hypothesis applied to $\mathcal{D}_0$, there is $\mathcal{D}'_0 \triangleright_{\text{Curry}} \Gamma, y : C, x : B \vdash s : D$. Hence, the derivation

$$\mathcal{D}' = \frac{\frac{\vdots \mathcal{D}'_0}{\Gamma, y : C, x : B \vdash s : D}}{\Gamma, x : B \vdash \lambda y.s : C \Rightarrow D} \lambda \quad \text{is such that } \mathcal{D}' \triangleright_{\text{Curry}} \Gamma, x : B \vdash t : A.$$

- If $r = @$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_1}{\Gamma \vdash t_1 : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}_2}{\Gamma \vdash t_2 : C}}{\Gamma \vdash t_1 t_2 : A} @$$

with $t = t_1 t_2$. By induction hypothesis applied to $\mathcal{D}_1$ and $\mathcal{D}_2$, there are $\mathcal{D}'_1 \triangleright_{\text{Curry}} \Gamma, x : B \vdash t_1 : C \Rightarrow A$

and $\mathcal{D}'_2 \triangleright_{\text{Curry}} \Gamma, x : B \vdash t_2 : C$. Hence, the derivation $\mathcal{D}' = \frac{\frac{\vdots \mathcal{D}'_1}{\Gamma, x : B \vdash t_1 : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}'_2}{\Gamma, x : B \vdash t_2 : C}}{\Gamma, x : B \vdash t_1 t_2 : A} @$

is such that $\mathcal{D}' \triangleright_{\text{Curry}} \Gamma, x : B \vdash t : A$.

2. Proof by induction on the size of a minimal derivation $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash t : A$. Let us consider the last rule r of $\mathcal{D}$. Cases:

- If $r = \text{var}$ then $\mathcal{D} = \overline{\Gamma', y : A \vdash y : A}^{\text{var}}$ with $\Gamma = \Gamma', y : A$ and $t = y$. Hence, $\text{fv}(t) = \{y\} \subseteq \text{dom}(\Gamma)$ and $\Gamma \upharpoonright_{\text{fv}(t)} = y : A$, so the derivation $\mathcal{D}' = \overline{\Gamma \upharpoonright_{\text{fv}(t)} \vdash y : A}^{\text{var}}$ is such that $\mathcal{D}' \triangleright_{\text{Curry}} \Gamma \upharpoonright_{\text{fv}(t)} \vdash t : A$.
- If $r = \lambda$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_0}{\Gamma, y : C \vdash s : D}}{\Gamma \vdash \lambda y.s : C \Rightarrow D} \lambda$$

with $A = C \Rightarrow D$ and $t = \lambda y.s$; we can suppose without loss of generality that $y \notin \text{dom}(\Gamma)$. By induction hypothesis applied to $\mathcal{D}_0$, $\text{fv}(s) \subseteq \text{dom}(\Gamma) \cup \{y\}$ and there is $\mathcal{D}'_0 \triangleright_{\text{Curry}} (\Gamma, y : C) \upharpoonright_{\text{fv}(s)} \vdash s : D$. Hence, $\text{fv}(t) = \text{fv}(s) \setminus \{y\} \subseteq \text{dom}(\Gamma)$, and the derivation

$$\mathcal{D}' = \frac{\frac{\vdots \mathcal{D}'_0}{(\Gamma, y : C) \upharpoonright_{\text{fv}(s)} \vdash s : D}}{\Gamma \upharpoonright_{\text{fv}(s)} \vdash \lambda y.s : C \Rightarrow D} \lambda$$

is such that $\mathcal{D}' \triangleright_{\text{Curry}} \Gamma \upharpoonright_{\text{fv}(t)} \vdash t : A$, since $y \notin \text{dom}(\Gamma) \supseteq \text{fv}(t)$.

- If $r = @$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_1}{\Gamma \vdash t_1 : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}_2}{\Gamma \vdash t_2 : C}}{\Gamma \vdash t_1 t_2 : A} @$$

with $t = t_1 t_2$. By induction hypothesis applied to $\mathcal{D}_1$ and $\mathcal{D}_2$, $\text{fv}(t_1) \subseteq \text{dom}(\Gamma)$ and $\text{fv}(t_2) \subseteq \text{dom}(\Gamma)$ and there are $\mathcal{D}'_1 \triangleright_{\text{Curry}} \Gamma \upharpoonright_{\text{fv}(t_1)} \vdash t_1 : C \Rightarrow A$ and $\mathcal{D}'_2 \triangleright_{\text{Curry}} \Gamma \upharpoonright_{\text{fv}(t_2)} \vdash t_2 : C$. Hence, $\text{fv}(t) = \text{fv}(t_1) \cup \text{fv}(t_2) \subseteq \text{dom}(\Gamma)$. By Item 1, there are $\mathcal{D}''_1 \triangleright_{\text{Curry}} \Gamma \upharpoonright_{\text{fv}(t)} \vdash t_1 : C \Rightarrow A$ and $\mathcal{D}''_2 \triangleright_{\text{Curry}} \Gamma \upharpoonright_{\text{fv}(t)} \vdash t_2 : C$. Therefore,

the derivation $\mathcal{D}' = \frac{\frac{\vdots \mathcal{D}''_1}{\Gamma \upharpoonright_{\text{fv}(t)} \vdash t_1 : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}''_2}{\Gamma \upharpoonright_{\text{fv}(t)} \vdash t_2 : C}}{\Gamma \upharpoonright_{\text{fv}(t)} \vdash t_1 t_2 : A} @$ is such that $\mathcal{D}' \triangleright_{\text{Curry}} \Gamma \upharpoonright_{\text{fv}(t)} \vdash t : A$. $\square$

Proposition. 1. (From Church to Curry) If $\Gamma \vdash_{\text{Church}} t : A$, then $\Gamma \vdash_{\text{Curry}} [t] : A$.

2. (From Curry to Church) If $\Gamma \vdash_{\text{Curry}} t : A$ then $\Gamma \vdash_{\text{Church}} t' : A$ for some t' in Church-style such that $[t'] = t$.

Proof. 1. Proof by induction on the size of a minimal derivation $\mathcal{D} \triangleright_{\text{Church}} \Gamma \vdash t : A$. Let us consider the last rule r of $\mathcal{D}$. Cases:

- If $r = \text{var}$ then $\mathcal{D} = \overline{\Gamma', y : A \vdash y : A}^{\text{var}}$ with $\Gamma = \Gamma', y : A$ and $t = y = [t]$. Hence $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash [t] : A$.
- If $r = \lambda$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_0}{\Gamma, y : C \vdash s : D}}{\Gamma \vdash \lambda y^C.s : C \Rightarrow D} \lambda$$

with $A = C \Rightarrow D$ and $t = \lambda y^C.s$; we can suppose without loss of generality that $y \notin \text{dom}(\Gamma)$. By induction hypothesis applied to $\mathcal{D}_0$, there is $\mathcal{D}'_0 \triangleright_{\text{Curry}} \Gamma, y : C \vdash [s] : D$. Hence, the derivation

$$\mathcal{D}' = \frac{\frac{\vdots \mathcal{D}'_0}{\Gamma, y : C \vdash [s] : D}}{\Gamma \vdash \lambda y.[s] : C \Rightarrow D} \lambda$$

is such that $\mathcal{D}' \triangleright_{\text{Curry}} \Gamma \vdash [t] : A$, since $[t] = \lambda y.[s]$.

- If $r = @$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_1}{\Gamma \vdash t_1 : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}_2}{\Gamma \vdash t_2 : C}}{\Gamma \vdash t_1 t_2 : A} @$$

with $t = t_1 t_2$. By induction hypothesis applied to $\mathcal{D}_1$ and $\mathcal{D}_2$, there are $\mathcal{D}'_1 \triangleright_{\text{Curry}} \Gamma \vdash [t_1] : C \Rightarrow A$

and $\mathcal{D}'_2 \triangleright_{\text{Curry}} \Gamma \vdash [t_2] : C$. Therefore, the derivation $\mathcal{D}' = \frac{\frac{\vdots \mathcal{D}'_1}{\Gamma \vdash t_1 : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}'_2}{\Gamma \vdash t_2 : C}}{\Gamma \vdash [t_1][t_2] : A} @$ is such that

$\mathcal{D}' \triangleright_{\text{Curry}} \Gamma \vdash [t] : A$, since $[t] = [t_1][t_2]$.

2. Proof by induction on the size of a minimal derivation $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash t : A$. Let us consider the last rule r of $\mathcal{D}$. Cases:

- If $r = \text{var}$ then $\mathcal{D} = \overline{\Gamma', y : A \vdash y : A}^{\text{var}}$ with $\Gamma = \Gamma', y : A$ and $t = y$. Let $t' := y$. Hence, $[t'] = y = t$ and $\mathcal{D} \triangleright_{\text{Church}} \Gamma \vdash t' : A$.
- If $r = \lambda$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_0}{\Gamma, y : C \vdash s : B}}{\Gamma \vdash \lambda y.s : C \Rightarrow B} \lambda$$

with $A = C \Rightarrow B$ and $t = \lambda y.s$; we can suppose without loss of generality that $y \notin \text{dom}(\Gamma)$. By induction hypothesis applied to $\mathcal{D}_0$, there is $\mathcal{D}'_0 \triangleright_{\text{Church}} \Gamma, y : C \vdash s' : B$ for some s' in Church style such that $[s'] = s$. Let $t' := \lambda y^C.s'$. Hence, the derivation

$$\mathcal{D}' = \frac{\frac{\vdots \mathcal{D}'_0}{\Gamma, y : C \vdash s' : B}}{\Gamma \vdash \lambda y^C.s' : C \Rightarrow B} \lambda$$

is such that $\mathcal{D}' \triangleright_{\text{Church}} \Gamma \vdash t' : A$, and $[t'] = \lambda y.[s'] = \lambda y.s = t$.

- If $r = @$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_1}{\Gamma \vdash t_1 : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}_2}{\Gamma \vdash t_2 : C}}{\Gamma \vdash t_1 t_2 : A} @$$

with $t = t_1 t_2$. By induction hypothesis applied to $\mathcal{D}_1$ and $\mathcal{D}_2$, there are $\mathcal{D}'_1 \triangleright_{\text{Church}} \Gamma \vdash t'_1 : C \Rightarrow A$ and $\mathcal{D}'_2 \triangleright_{\text{Church}} \Gamma \vdash t'_2 : C$ for some t'_1 and t'_2 in Church style such that $[t'_1] = t_1$ and $[t'_2] = t_2$. Let

$t' := t'_1 t'_2$. Thus, the derivation $\mathcal{D}' = \frac{\frac{\vdots \mathcal{D}'_1}{\Gamma \vdash t'_1 : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}'_2}{\Gamma \vdash t'_2 : C}}{\Gamma \vdash t'_1 t'_2 : A} @$ is such that $\mathcal{D}' \triangleright_{\text{Church}} \Gamma \vdash t' : A$,
and $\lceil t' \rceil = \lceil t'_1 \rceil \lceil t'_2 \rceil = t_1 t_2 = t$. $\square$

Proposition (Uniqueness). *If $\mathcal{D} \triangleright_{\text{Church}} \Gamma \vdash t : A$ and $\mathcal{D}' \triangleright_{\text{Church}} \Gamma \vdash t : A'$, then $A = A'$ and $\mathcal{D} = \mathcal{D}'$.*

Proof. By induction on t . Cases:

- If $t = y$ then $\mathcal{D} = \overline{\Gamma_0, y : A \vdash y : A}^{\text{var}} = \mathcal{D}'$ with $\Gamma = \Gamma_0, y : A$ and $t = y$.
- If $t = \lambda x. t_0$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_0}{\Gamma, y : C \vdash t_0 : B}}{\Gamma \vdash \lambda y^{C.t_0} : C \Rightarrow B} \lambda \quad \mathcal{D}' = \frac{\frac{\vdots \mathcal{D}'_0}{\Gamma, y : C' \vdash t_0 : B'}}{\Gamma \vdash \lambda y^{C'.t_0} : C' \Rightarrow B'} \lambda$$

with $A = C \Rightarrow B$ and $A' = C' \Rightarrow B'$; we can suppose without loss of generality that $y \notin \text{dom}(\Gamma)$. As $\lambda y^{C.t_0} = t = \lambda y^{C'.t_0}$, then $C = C'$. By induction hypothesis applied to t_0 , $B = B'$ and $\mathcal{D}_0 = \mathcal{D}'_0$. Hence, $A = C \Rightarrow B = C' \Rightarrow B' = A'$ and so $\mathcal{D} = \mathcal{D}'$.

- If $t = t_1 t_2$ then

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_1}{\Gamma \vdash t_1 : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}_2}{\Gamma \vdash t_2 : C}}{\Gamma \vdash t_1 t_2 : A} @ \quad \mathcal{D}' = \frac{\frac{\vdots \mathcal{D}'_1}{\Gamma \vdash t_1 : C' \Rightarrow A'} \quad \frac{\vdots \mathcal{D}'_2}{\Gamma \vdash t_2 : C'}}{\Gamma \vdash t_1 t_2 : A'} @$$

By induction hypothesis applied to t_1 and t_2 , $C \Rightarrow A = C' \Rightarrow A'$, $C = C'$, $\mathcal{D}_1 = \mathcal{D}'_1$ and $\mathcal{D}_2 = \mathcal{D}'_2$. Therefore, $A = A'$ and $\mathcal{D} = \mathcal{D}'$. $\square$

Lemma (Substitution). *If $\Gamma, x : B \vdash_{\text{STLC}} t : A$ and $\Gamma \vdash_{\text{STLC}} s : B$, then $\Gamma \vdash_{\text{STLC}} t\{s/x\} : A$.*

Proof. We prove both statement for STLC in Curry style. The proof for Church style is analogous. Proof by structural induction t . Cases:

- If $t = x$ then $t\{s/x\} = s$. As $\Gamma, x : B \vdash_{\text{Curry}} t : A$, then $x \notin \text{dom}(\Gamma)$ and $A = B$. Hence, the hypothesis $\Gamma \vdash_{\text{Curry}} s : B$ means that $\Gamma \vdash_{\text{Curry}} t\{s/x\} : A$.
- If $t = y \neq x$ then $t\{s/x\} = y = t$. As $\Gamma, x : B \vdash_{\text{Curry}} t : A$, then $\Gamma(y) = A$ and so $\Gamma \vdash_{\text{Curry}} t\{s/x\} : A$ by applying the var rule.
- If $t = \lambda y. t_0$ then we can suppose without loss of generality that $y \notin \text{fv}(s) \cup \{x\}$, thus $t\{s/x\} = \lambda y. (t_0\{s/x\})$. As $\Gamma, x : B \vdash_{\text{Curry}} t : A$, then $A = C \Rightarrow D$ and $\Gamma, x : B, y : C \vdash_{\text{Curry}} t_0 : D$. By induction hypothesis applied to t_0 , there is a derivation $\mathcal{D}_0 \triangleright_{\text{Curry}} \Gamma, y : C \vdash_{\text{Curry}} t_0\{s/x\} : D$. Therefore, the derivation

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_0}{\Gamma, y : C \vdash t_0\{s/x\} : D}}{\Gamma \vdash \lambda y. (t_0\{s/x\}) : C \Rightarrow D} \lambda$$

is such that $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash t\{s/x\} : A$.

- If $t = t_1 t_2$ then $t\{s/x\} = t_1\{s/x\} t_2\{s/x\}$. As $\Gamma, x : B \vdash_{\text{Curry}} t : A$, then $\Gamma, x : B \vdash_{\text{Curry}} t_1 : C \Rightarrow A$ and $\Gamma, x : B \vdash_{\text{Curry}} t_2 : C$. By induction hypothesis applied to t_1 and t_2 , there are derivations $\mathcal{D}_1 \triangleright_{\text{Curry}} \Gamma \vdash_{\text{Curry}} t_1\{s/x\} : C \Rightarrow A$ and $\mathcal{D}_2 \triangleright_{\text{Curry}} \Gamma \vdash_{\text{Curry}} t_2\{s/x\} : C$. Therefore, the derivation

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_1}{\Gamma \vdash t_1\{s/x\} : C \Rightarrow A} \quad \frac{\vdots \mathcal{D}_2}{\Gamma \vdash t_2\{s/x\} : C}}{\Gamma \vdash t_1\{s/x\} t_2\{s/x\} : A} @$$

is such that $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash t\{s/x\} : A$. $\square$

Theorem (Subject reduction). *If $\Gamma \vdash_{\text{STLC}} t : A$ and $t \rightarrow_{\beta} s$, then $\Gamma \vdash_{\text{STLC}} s : A$.*

Proof. We prove both statement for STLC in Curry style. The proof for Church style is analogous. Proof by structural induction on t . Cases:

- If $t = x$ then there is no s such that $t \rightarrow_{\beta} s$, and the statement is vacuously true.
- If $t = \lambda x.t_0$, then $A = B \Rightarrow C$ and we can suppose without loss of generality that $x \notin \text{dom}(\Gamma)$. Thus, $t \rightarrow_{\beta} \lambda x.s_0 = s$ with $t_0 \rightarrow_{\beta} s_0$, and $\Gamma, x : B \vdash_{\text{Curry}} t_0 : C$. By induction hypothesis applied to t_0 , there is a derivation $\mathcal{D}_0 \triangleright_{\text{Curry}} \Gamma, x : B \vdash s_0 : C$. Therefore, the derivation

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_0}{\Gamma, x : B \vdash s_0 : C}}{\Gamma \vdash \lambda x.s_0 : B \Rightarrow C}^{\lambda}$$

is such that $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash s : A$.

- If $t = t_1 t_2$, then $\Gamma \vdash_{\text{Curry}} t_1 : B \Rightarrow A$ and $\Gamma \vdash_{\text{Curry}} t_2 : B$. There are three subcases:
 - If $t_1 = \lambda x.t'_1$ (we can suppose without loss of generality that $x \notin \text{dom}(\Gamma)$) and $t \rightarrow_{\beta} t'_1\{t_2/x\} = s$, then $\Gamma, x : B \vdash_{\text{Curry}} t'_1 : A$. By the substitution lemma above, $\Gamma \vdash_{\text{Curry}} t'_1\{t_2/x\} : A$ and so $\Gamma \vdash_{\text{Curry}} s : A$.
 - If $t \rightarrow_{\beta} t'_1 t_2 = s$ with $t_1 \rightarrow_{\beta} t'_1$, then by induction hypothesis applied to t_1 , there is a derivation $\mathcal{D}_1 \triangleright_{\text{Curry}} \Gamma \vdash t'_1 : B \Rightarrow A$. Therefore, the derivation

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_1}{\Gamma \vdash t'_1 : B \Rightarrow A} \quad \frac{\vdots}{\Gamma \vdash t_2 : B}}{\Gamma \vdash t'_1 t_2 : A}^{\textcircled{A}}$$

is such that $\mathcal{D} \triangleright_{\text{Curry}} \Gamma \vdash s : A$.

- If $t \rightarrow_{\beta} t_1 t'_2 = s$ with $t_2 \rightarrow_{\beta} t'_2$, then the proof is analogous to the previous subcase. □