

The λ -calculus, typed and untyped

<https://cl-teaching.uibk.ac.at/26w/lc/>

Solutions to selected exercises from Lecture 1

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Assessed exercises from Lecture 1

(<https://cl-teaching.uibk.ac.at/26w/lc/exercises/01.pdf>)

Exercise 1 (3 points)

Prove that $(X \Rightarrow Y) \Rightarrow X \vdash (X \Rightarrow Y) \Rightarrow Y$ by using both ND and ND_{seq}.

Solution to Exercise 1

In ND:

$$\frac{\frac{[X \Rightarrow Y]^*}{Y} \quad \frac{(X \Rightarrow Y) \Rightarrow X \quad [X \Rightarrow Y]^*}{X} \Rightarrow_e}{(X \Rightarrow Y) \Rightarrow Y} \Rightarrow_i^*$$

In ND_{seq}:

$$\frac{\frac{(X \Rightarrow Y) \Rightarrow X, X \Rightarrow Y \vdash X \Rightarrow Y}{} \text{ax} \quad \frac{\frac{(X \Rightarrow Y) \Rightarrow X, X \Rightarrow Y \vdash (X \Rightarrow Y) \Rightarrow X}{} \text{ax} \quad \frac{(X \Rightarrow Y) \Rightarrow X, X \Rightarrow Y \vdash X \Rightarrow Y}{} \text{ax}}{(X \Rightarrow Y) \Rightarrow X, X \Rightarrow Y \vdash X} \Rightarrow_e}{(X \Rightarrow Y) \Rightarrow X, X \Rightarrow Y \vdash Y} \Rightarrow_e \Rightarrow_i$$

Exercise 2 (2 points)

Let us extend minimal logic to conjunction. To this end, consider the language defined by the BNF grammar below

$$A, B, C ::= X \mid A \Rightarrow B \mid A \wedge B$$

where $\wedge$ stands for the conjunction (“and”). Consider the system ND ^{$\wedge$} obtained by adding the inference rules below to ND.

$$\frac{A \quad B}{A \wedge B} \wedge_i \qquad \frac{A \wedge B}{A} \wedge_{e1} \qquad \frac{A \wedge B}{B} \wedge_{e2}$$

Define the corresponding inference rules for the system ND_{seq} ^{$\wedge$} , that is, define the version of the inference rules above that is compatible with the ND_{seq} formalism.

Solution to Exercise 2

The inference rules of ND_{seq} ^{$\wedge$} are the ones of ND_{seq} plus the following ones:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_i \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge_{e1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge_{e2}$$

Exercise 3 (5 points)

Define the notions of $\wedge$ -cut-formula and $\wedge$ -redex in the system $\text{ND}^\wedge$ concerning the rules $\wedge_i$, $\wedge_{e_1}$ and $\wedge_{e_2}$ from Exercise 2. Define the cut-elimination steps for a $\wedge$ -redex. State and prove the cut elimination theorem for $\text{ND}^\wedge$.

Hint. Get inspired from the corresponding notions, theorems and proofs for ND seen in Lecture 1.

Solution to Exercise 3

A $\wedge$ -cut-formula in a derivation $\mathcal{D}$ in $\text{ND}^\wedge$ is a formula occurrence in $\mathcal{D}$ that is both the conclusion of a $\wedge_i$ rule and the premise of a $\wedge_{e_1}$ or $\wedge_{e_2}$ rule.

A $\wedge$ -redex in a derivation $\mathcal{D}$ in $\text{ND}^\wedge$ is a pair of instances of $\Rightarrow_i$ and $\Rightarrow_e$ in $\mathcal{D}$ containing a $\wedge$ -cut-formula.

The cut-elimination steps for the a $\wedge$ -redex are the following:

$$\begin{array}{ccc} \begin{array}{c} \vdots \mathcal{D}_1 \\ A_1 \\ \hline A_1 \wedge A_2 \\ \hline A_1 \end{array} \wedge_i & \xrightarrow{\text{cut}} & \begin{array}{c} \vdots \mathcal{D}_1 \\ A_1 \end{array} \\ \begin{array}{c} \vdots \mathcal{D}_2 \\ A_2 \\ \hline A_1 \wedge A_2 \\ \hline A_2 \end{array} \wedge_{e_1} & & \end{array} \quad \begin{array}{ccc} \begin{array}{c} \vdots \mathcal{D}_1 \\ A_1 \\ \hline A_1 \wedge A_2 \\ \hline A_2 \end{array} \wedge_i & \xrightarrow{\text{cut}} & \begin{array}{c} \vdots \mathcal{D}_2 \\ A_2 \end{array} \\ \begin{array}{c} \vdots \mathcal{D}_2 \\ A_2 \\ \hline A_1 \wedge A_2 \\ \hline A_2 \end{array} \wedge_{e_2} & & \end{array}$$

To prove the cut-elimination theorem, we first introduce some notions and notations.

1. A *redex* is a $\wedge$ -redex or a $\Rightarrow$ -redex, and a *cut-formula* is $\wedge$ -cut formula or a $\Rightarrow$ -cut formula.
2. The *size* of a cut-formula is the number of connectives ($\Rightarrow$ and $\wedge$) it contains.
3. The *weight* of a redex is the size of its cut-formula. The *weight* of a derivation $\mathcal{D}$ in $\text{ND}^\wedge$ is the finite multiset of the weights of the redexes in $\mathcal{D}$.
4. We write $\mathcal{D} \triangleright_{\text{ND}^\wedge} \Gamma \vdash A$ if $\mathcal{D}$ is a derivation in $\text{ND}^\wedge$ with conclusion A and hypotheses among the formulas in Γ .

Theorem (Cut-elimination, aka normalization, in $\text{ND}^\wedge$). *If $\mathcal{D} \triangleright_{\text{ND}^\wedge} \Gamma \vdash A$, then there is $\mathcal{D}' \triangleright_{\text{ND}^\wedge} \Gamma \vdash A$ without redexes such that $\mathcal{D} \xrightarrow{*}_{\text{cut}} \mathcal{D}'$.*

Proof. If $\mathcal{D}$ is without redexes, we are done. Otherwise, take a redex r in $\mathcal{D}$ such that there are no redexes above the elimination rule ($\Rightarrow_e$ or $\wedge_{e_1}$ or $\wedge_{e_2}$) in r ; such a r exists because $\mathcal{D}$ is finite! Apply $\rightarrow_{\text{cut}}$ to r to get $\mathcal{D}_1 \triangleright_{\text{ND}^\wedge} \Gamma \vdash A$ where redexes are not duplicated (as r is an uppermost redex), new redexes can be created but have a lower weight (smaller cut-formula). Therefore, $w(\mathcal{D}) \succ_{\text{mul}} w(\mathcal{D}_1)$. By induction hypothesis on the weight of derivations, we conclude. $\square$

Remark. The proof of cut-elimination for $\text{ND}^\wedge$ is literally the same as the one for ND (see Lecture 1). What is new in this proof is the meaning of the expressions “cut-formula”, “redex”, “size of a formula” and so on. Those notions have been defined (see above) so as to allow the proof of cut-elimination of ND to adapt smoothly to $\text{ND}^\wedge$.

Non-assessed exercises from Lecture 1

(<https://cl-teaching.uibk.ac.at/26w/lc/slides/01.pdf>)

Exercise 1

Prove the following facts, using ND and ND_{seq} .

1. $\vdash X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)$.
2. $(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z$.
3. $(X \Rightarrow Y) \Rightarrow X \vdash Y \Rightarrow X$.
4. $X \Rightarrow (Y \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z$.
5. $X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y \vdash X \Rightarrow Z$.
6. $(X \Rightarrow X) \Rightarrow Y \vdash (Y \Rightarrow Z) \Rightarrow Z$.

Solution to Exercise 1

1. In ND and ND_{seq}, respectively:

$$\frac{\frac{\frac{[X \Rightarrow Y]^\circ \quad [X]^*}{\Rightarrow_e} \quad Y}{(X \Rightarrow Y) \Rightarrow Y} \Rightarrow_i^\circ}{X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)} \Rightarrow_i^*$$

$$\frac{\frac{\frac{\frac{\overline{X, X \Rightarrow Y \vdash X \Rightarrow Y}^{\text{ax}} \quad \overline{X, X \Rightarrow Y \vdash X}^{\text{ax}}}{\Rightarrow_e} \quad X, X \Rightarrow Y \vdash Y}{X \vdash (X \Rightarrow Y) \Rightarrow Y} \Rightarrow_i}{\vdash X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)} \Rightarrow_i$$

2. In ND:

$$\frac{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z) \quad \frac{[Y]^*}{X \Rightarrow Y} \Rightarrow_i}{\frac{X \Rightarrow Z}{Y \Rightarrow X \Rightarrow Z} \Rightarrow_i^*} \Rightarrow_e$$

In ND_{seq}:

$$\frac{\frac{\overline{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z), Y \vdash (X \Rightarrow Y) \Rightarrow (X \Rightarrow Z)}^{\text{ax}} \quad \frac{\overline{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z), X, Y \vdash Y}^{\text{ax}}}{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z), Y \vdash X \Rightarrow Y} \Rightarrow_i}{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z), Y \vdash X \Rightarrow Z} \Rightarrow_e}{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z} \Rightarrow_i^*$$

3. In ND and ND_{seq}, respectively:

$$\frac{(X \Rightarrow Y) \Rightarrow X \quad \frac{[Y]^*}{X \Rightarrow Y} \Rightarrow_i}{\frac{X}{Y \Rightarrow X} \Rightarrow_i^*} \Rightarrow_e$$

$$\frac{\overline{(X \Rightarrow Y) \Rightarrow X, Y \vdash (X \Rightarrow Y) \Rightarrow X}^{\text{ax}} \quad \frac{\overline{(X \Rightarrow Y) \Rightarrow X, X, Y \vdash Y}^{\text{ax}}}{(X \Rightarrow Y) \Rightarrow X, Y \vdash X \Rightarrow Y} \Rightarrow_i}{\frac{(X \Rightarrow Y) \Rightarrow X, Y \vdash X}{(X \Rightarrow Y) \Rightarrow X \vdash Y \Rightarrow X} \Rightarrow_i} \Rightarrow_e$$

4. In ND:

$$\frac{\frac{X \Rightarrow (Y \Rightarrow Z) \quad [X]^\circ}{Y \Rightarrow Z} \Rightarrow_e \quad \frac{[Y]^*}{\frac{Z}{X \Rightarrow Z} \Rightarrow_i^\circ} \Rightarrow_e}{\frac{X \Rightarrow Z}{Y \Rightarrow X \Rightarrow Z} \Rightarrow_i^*} \Rightarrow_e$$

In ND_{seq}:

$$\frac{\overline{X \Rightarrow (Y \Rightarrow Z), X, Y \vdash X \Rightarrow (Y \Rightarrow Z)}^{\text{ax}} \quad \overline{X \Rightarrow (Y \Rightarrow Z), X, Y \vdash X}^{\text{ax}}}{\frac{X \Rightarrow (Y \Rightarrow Z), X, Y \vdash Y \Rightarrow Z}{\frac{X \Rightarrow (Y \Rightarrow Z), X, Y \vdash Z}{\frac{X \Rightarrow (Y \Rightarrow Z), Y \vdash X \Rightarrow Z}{X \Rightarrow (Y \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z} \Rightarrow_i} \Rightarrow_i} \Rightarrow_e$$

5. In ND:

$$\frac{\frac{X \Rightarrow Y \Rightarrow Z \quad [X]^*}{Y \Rightarrow Z} \Rightarrow_e \quad \frac{X \Rightarrow Y \quad [X]^*}{Y} \Rightarrow_e}{\frac{Z}{X \Rightarrow Z} \Rightarrow_i^*} \Rightarrow_e$$

In ND_{seq}:

$$\frac{\frac{\frac{\frac{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X \Rightarrow Y \Rightarrow Z}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X}^{\text{ax}} \quad \frac{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X \Rightarrow Y}^{\text{ax}} \quad \frac{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X \Rightarrow Y}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X}^{\text{ax}} \quad \frac{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X}^{\text{ax}}}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash Y \Rightarrow Z}^{\Rightarrow_e} \quad \frac{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash Y}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash Y}^{\Rightarrow_e}}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash Z}^{\Rightarrow_e} \quad \frac{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash Z}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y \vdash X \Rightarrow Z}^{\Rightarrow_i}$$

6. In ND:

$$\frac{\frac{[Y \Rightarrow Z]^*}{Y} \quad \frac{\frac{(X \Rightarrow X) \Rightarrow Y}{X \Rightarrow X}^{\Rightarrow_i^o} \quad \frac{[X]^o}{X \Rightarrow X}^{\Rightarrow_i^o}}{Y}^{\Rightarrow_e}}{Z}^{\Rightarrow_e} \quad \frac{Z}{(Y \Rightarrow Z) \Rightarrow Z}^{\Rightarrow_i^*}$$

In ND_{seq}:

$$\frac{\frac{\frac{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash Y \Rightarrow Z}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash (X \Rightarrow X) \Rightarrow Y}^{\text{ax}} \quad \frac{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash (X \Rightarrow X) \Rightarrow Y}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash X \Rightarrow X}^{\text{ax}} \quad \frac{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash X \Rightarrow X}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash X \Rightarrow X}^{\text{ax}}}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash Y}^{\Rightarrow_e} \quad \frac{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash Y}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash Y}^{\Rightarrow_e}}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash Y}^{\Rightarrow_e} \quad \frac{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash Y}{(X \Rightarrow X) \Rightarrow Y \vdash (Y \Rightarrow Z) \Rightarrow Z}^{\Rightarrow_i}$$

Exercise 2

Show that $\not\vdash (X \Rightarrow Y) \Rightarrow X$, i.e. $(X \Rightarrow Y) \Rightarrow X$ is not derivable with no hypotheses.

Solution to Exercise 2

Suppose by absurd that $(X \Rightarrow Y) \Rightarrow X$ is derivable in ND with no hypothesis. The last rule of the derivation cannot be either an hypothesis (because there are no hypotheses) or $\Rightarrow_e$ (otherwise it would be it would contradict the subformula property), hence it could only be $\Rightarrow_i$ discharging the hypothesis $X \Rightarrow Y$, that is,

$$\frac{\frac{[X \Rightarrow Y]^*}{\vdots} \quad X}{(X \Rightarrow Y) \Rightarrow X}^{\Rightarrow_i^*}$$

The rule whose conclusion is X cannot be either $\Rightarrow_i$ (otherwise its conclusion should be an arrow) or an hypothesis (because there is no hypothesis X), hence it could only be $\Rightarrow_e$ with premises $A \Rightarrow X$ and A for some formula A , that is,

$$\frac{\frac{[X \Rightarrow Y]^*}{\vdots} \quad A \Rightarrow X \quad \frac{[X \Rightarrow Y]^*}{\vdots} \quad A}{X}^{\Rightarrow_e} \quad \frac{X}{(X \Rightarrow Y) \Rightarrow X}^{\Rightarrow_i^*}$$

For the subformula property applied to the derivation whose conclusion is X , A could only be a subformula of X or $X \Rightarrow Y$, that is,

- either $A = X$, but then $A \Rightarrow X = X \Rightarrow X$ is a formula of that derivation that is not a subformula of X or $X \Rightarrow Y$, which contradicts the subformula property;
- or $A = Y$, but then $A \Rightarrow X = Y \Rightarrow X$ is a formula of that derivation that is not a subformula of X or $X \Rightarrow Y$, which contradicts the subformula property;
- or $A = X \Rightarrow Y$, but then $A \Rightarrow X = (X \Rightarrow Y) \Rightarrow X$ is a formula of that derivation that is not a subformula of X or $X \Rightarrow Y$, which contradicts the subformula property.

Therefore, there is no derivation of $(X \Rightarrow Y) \Rightarrow X$ with no hypotheses.

Exercise 3

Perform all possible cut-elimination steps from the derivation on p. 24 of Day 1 slides, until you get a derivation without redexes. Is it always the same?

Solution to Exercise 3

The derivation on p. 24 of Day 1 slides is $\mathcal{D}$ below, where there are two redexes, marked as **blue** and **red**.

$$\begin{array}{c}
 \frac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^\dagger \quad [X \Rightarrow X]^\circ}{B \Rightarrow (X \Rightarrow X)} \Rightarrow_e \quad \frac{[(X \Rightarrow X) \Rightarrow B]^* \quad [X \Rightarrow X]^\circ}{B} \Rightarrow_e \\
 \frac{\frac{X \Rightarrow X}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)} \Rightarrow_{i^\circ} \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{\frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^*} \Rightarrow_e \\
 \frac{\frac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^\dagger \quad [X \Rightarrow X]^\circ}{B \Rightarrow (X \Rightarrow X)} \Rightarrow_e \quad \frac{[(X \Rightarrow X) \Rightarrow B]^* \quad [X \Rightarrow X]^\circ}{B} \Rightarrow_e}{\frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^*} \Rightarrow_e \\
 \frac{\frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^* \quad \frac{[X \Rightarrow X]^\dagger}{B \Rightarrow X \Rightarrow X} \Rightarrow_i}{\frac{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)} \Rightarrow_i^\dagger} \Rightarrow_e \\
 \frac{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_e
 \end{array}$$

If the **red** redex in $\mathcal{D}$ is fired, then $\mathcal{D}$ reduces to the derivation $\mathcal{D}_1$ below.

$$\begin{array}{c}
 \frac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^\dagger \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{B \Rightarrow (X \Rightarrow X)} \Rightarrow_e \quad \frac{[(X \Rightarrow X) \Rightarrow B]^* \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{B} \Rightarrow_e \\
 \frac{\frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^* \quad \frac{[X \Rightarrow X]^\dagger}{B \Rightarrow X \Rightarrow X} \Rightarrow_i}{\frac{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)} \Rightarrow_i^\dagger} \Rightarrow_e \\
 \frac{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_e
 \end{array}$$

If the **blue** redex in $\mathcal{D}_1$ is fired, then $\mathcal{D}_1$ reduces to the derivation $\mathcal{D}'_1$ below, with a new **green** redex.

$$\begin{array}{c}
 \frac{[X \Rightarrow X]^\dagger}{B \Rightarrow X \Rightarrow X} \Rightarrow_i \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet \quad \frac{[(X \Rightarrow X) \Rightarrow B]^* \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{B} \Rightarrow_e \\
 \frac{\frac{[X \Rightarrow X]^\dagger}{B \Rightarrow X \Rightarrow X} \Rightarrow_i \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)} \Rightarrow_i^\dagger \quad \frac{[(X \Rightarrow X) \Rightarrow B]^* \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{B} \Rightarrow_e \\
 \frac{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X) \quad B}{B \Rightarrow (X \Rightarrow X)} \Rightarrow_e \\
 \frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^*
 \end{array}$$

If the **green** redex in $\mathcal{D}'_1$ is fired, then $\mathcal{D}'_1$ reduces to derivation $\mathcal{D}''_1$ below, with a new **gray** redex.

$$\begin{array}{c}
 \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet \quad \frac{[(X \Rightarrow X) \Rightarrow B]^* \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{B} \Rightarrow_e \\
 \frac{B \Rightarrow X \Rightarrow X \quad B}{B \Rightarrow (X \Rightarrow X)} \Rightarrow_e \\
 \frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^*
 \end{array}$$

If the **gray** redex in $\mathcal{D}''_1$ is fired, then $\mathcal{D}''_1$ reduces to derivation $\mathcal{D}_0$ below, which is without redexes.

$$\frac{\frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i$$

If the **blue** redex in $\mathcal{D}$ is fired, then $\mathcal{D}$ reduces to the derivation $\mathcal{D}_2$ below, with a new **green** redex.

$$\frac{\frac{\frac{[X \Rightarrow X]^\dagger}{B \Rightarrow X \Rightarrow X} \Rightarrow_i}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)} \Rightarrow_i^\dagger \quad \frac{[X \Rightarrow X]^\circ \quad [(X \Rightarrow X) \Rightarrow B]^* \quad [X \Rightarrow X]^\circ}{B} \Rightarrow_e}{B \Rightarrow (X \Rightarrow X)} \Rightarrow_e$$

$$\frac{\frac{\frac{X \Rightarrow X}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^\circ \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{X \Rightarrow X} \Rightarrow_e}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^*$$

If the **red** redex in $\mathcal{D}_2$ is fired, then $\mathcal{D}_2$ reduces to the derivation $\mathcal{D}_{21}$ below.

$$\frac{\frac{\frac{[X \Rightarrow X]^\dagger}{B \Rightarrow X \Rightarrow X} \Rightarrow_i \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)} \Rightarrow_i^\dagger \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{B \Rightarrow (X \Rightarrow X)} \Rightarrow_e \quad \frac{[(X \Rightarrow X) \Rightarrow B]^* \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{B} \Rightarrow_e}{\frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^*} \Rightarrow_e$$

If the **green** redex in $\mathcal{D}_{21}$ is fired, then $\mathcal{D}_{21}$ reduces to the derivation $\mathcal{D}_1''$ already shown above.

If the **green** redex in $\mathcal{D}_2$ is fired, then $\mathcal{D}_2$ reduces to the derivation $\mathcal{D}_{22}$ below, with a new gray redex.

$$\frac{\frac{[X \Rightarrow X]^\circ}{B \Rightarrow X \Rightarrow X} \Rightarrow_i \quad \frac{[(X \Rightarrow X) \Rightarrow B]^* \quad [X \Rightarrow X]^\circ}{B} \Rightarrow_e}{X \Rightarrow X} \Rightarrow_e \quad \frac{[X]^\bullet}{X \Rightarrow X} \Rightarrow_i^\bullet}{\frac{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}{X \Rightarrow X} \Rightarrow_i^\circ} \Rightarrow_e \quad \frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^*$$

If the **red** redex in $\mathcal{D}_{22}$ is fired, then $\mathcal{D}_{22}$ reduces to the derivation $\mathcal{D}_{221}$ below.

$$\frac{\frac{\frac{[X]^{\bullet}}{X \Rightarrow X} \Rightarrow_i^{\bullet}}{B \Rightarrow X \Rightarrow X} \Rightarrow_i}{\frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^*} \frac{[(X \Rightarrow X) \Rightarrow B]^* \frac{[X]^{\bullet}}{X \Rightarrow X} \Rightarrow_i^{\bullet}}{B} \Rightarrow_e$$

If the gray redex in $\mathcal{D}_{221}$ is fired, then $\mathcal{D}_{221}$ reduces to the derivation $\mathcal{D}_0$ below, which is without redexes.

$$\frac{\frac{[X]^{\bullet}}{X \Rightarrow X} \Rightarrow_i^{\bullet}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i$$

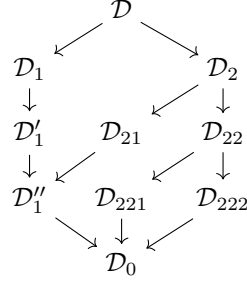
If the gray redex in $\mathcal{D}_{22}$ is fired, then $\mathcal{D}_{22}$ reduces to the derivation $\mathcal{D}_{222}$ below.

$$\frac{\frac{[X \Rightarrow X]^{\circ}}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^{\circ} \quad \frac{[X]^{\bullet}}{X \Rightarrow X} \Rightarrow_i^{\bullet}}{X \Rightarrow X} \Rightarrow_e \quad \frac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i^*$$

If the **red** redex in $\mathcal{D}_{222}$ is fired, then $\mathcal{D}_{222}$ reduces to the derivation $\mathcal{D}_0$ below, which is without redexes.

$$\frac{\frac{[X]^{\bullet}}{X \Rightarrow X} \Rightarrow_i^{\bullet}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)} \Rightarrow_i$$

All possible cut-elimination steps from $\mathcal{D}$ are the following:



In any case, every reduction sequence eventually reaches the same derivation $\mathcal{D}_0$ with no redexes.

Exercise 4

Order the following multisets over $\mathbb{N}$ according to the (strict) multiset order $\prec_{\text{mul}}$.

$$[1, 1] \quad [0, 2] \quad [1] \quad [0, 0, 2] \quad [] \quad [0, 3] \quad [0, 2, 2]$$

Solution to Exercise 4

$$[] \prec_{\text{mul}} [1] \prec_{\text{mul}} [1, 1] \prec_{\text{mul}} [0, 2] \prec_{\text{mul}} [0, 0, 2] \prec_{\text{mul}} [0, 2, 2] \prec_{\text{mul}} [0, 3].$$

Exercise 5

Prove in a rigorous way the proposition on p. 15 of Day 1 slides.

Solution to Exercise 5

Proposition. Let Γ be a finite multiset of formulas and A be a formula: $\Gamma \vdash A$ in ND if and only if the sequent $\Gamma \vdash A$ is derivable in ND_{seq} .

Proof. $\Rightarrow$: By induction on the number of rules of the smallest derivation $\mathcal{D}$ in ND proving that $\Gamma \vdash A$. Cases:

- $\mathcal{D}$ is just an hypothesis, that is, $\mathcal{D} = A$ and so $\Gamma = \Gamma', A$ for any finite multiset Γ' . Then, the derivation $\mathcal{D}_{\text{seq}}$ below derives the sequent $\Gamma \vdash A$ in ND_{seq} .

$$\mathcal{D}_{\text{seq}} = \frac{}{\Gamma', A \vdash A} \text{ax}$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_i$, that is, $A = B \Rightarrow C$ and

$$\mathcal{D} = \frac{\begin{array}{c} [B]^* \\ \vdots \\ \mathcal{D}' \\ C \end{array}}{B \Rightarrow C} \Rightarrow_i^*$$

where $\mathcal{D}'$ is the smallest derivation in ND that proves that $\Gamma, B \vdash C$, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$, there is a derivation $\mathcal{D}'_{\text{seq}}$ in ND_{seq} of the sequent $\Gamma, B \vdash C$. Then, the derivation $\mathcal{D}_{\text{seq}}$ below derives the sequent $\Gamma \vdash A$ in ND_{seq} .

$$\mathcal{D}_{\text{seq}} = \frac{\begin{array}{c} \vdots \\ \mathcal{D}'_{\text{seq}} \\ \Gamma, B \vdash C \end{array}}{\Gamma \vdash B \Rightarrow C} \Rightarrow_i$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_e$, that is, for some formula B

$$\mathcal{D} = \frac{\begin{array}{c} \vdots \\ \mathcal{D}' \\ B \Rightarrow A \end{array} \quad \begin{array}{c} \vdots \\ \mathcal{D}'' \\ B \end{array}}{A} \Rightarrow_e$$

where $\mathcal{D}'$ and $\mathcal{D}''$ are the smallest derivation in ND that prove that $\Gamma \vdash B \Rightarrow A$ and $\Gamma \vdash B$, respectively, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$ and $\mathcal{D}''$, respectively, there are derivations $\mathcal{D}'_{\text{seq}}$ and $\mathcal{D}''_{\text{seq}}$ in ND_{seq} of the sequents $\Gamma \vdash B \Rightarrow A$ and $\Gamma \vdash B$. Then, the derivation $\mathcal{D}_{\text{seq}}$ below derives the sequent $\Gamma \vdash A$ in ND_{seq} .

$$\mathcal{D}_{\text{seq}} = \frac{\frac{\vdots \mathcal{D}'_{\text{seq}}}{\Gamma \vdash B \Rightarrow A} \quad \frac{\vdots \mathcal{D}''_{\text{seq}}}{\Gamma \vdash B}}{\Gamma \vdash A} \Rightarrow_e$$

$\Leftarrow$: By induction on the number of rules of the smallest derivation $\mathcal{D}$ in ND_{seq} proving the sequent $\Gamma \vdash A$. Cases:

- The last rule of $\mathcal{D}$ is ax , that is,

$$\mathcal{D} = \frac{}{\Gamma', A \vdash A} \text{ax}$$

where $\Gamma = \Gamma', A$ for some finite multiset Γ' . Then, the derivation $\mathcal{D}_0 = A$ proves that $\Gamma \vdash A$ in ND .

- The last rule in $\mathcal{D}$ is $\Rightarrow_i$, that is, $A = B \Rightarrow C$ and

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}'}{\Gamma, B \vdash C}}{\Gamma \vdash B \Rightarrow C} \Rightarrow_i$$

where $\mathcal{D}'$ is the smallest derivation in ND_{seq} of the sequent $\Gamma, B \vdash C$, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$, there is a derivation $\mathcal{D}'_0$ in ND that proves that $\Gamma, B \vdash C$. Then, the derivation $\mathcal{D}_0$ below proves that $\Gamma \vdash A$ in ND .

$$\mathcal{D} = \frac{\frac{[B]^*}{\vdots \mathcal{D}'} \quad C}{B \Rightarrow C} \Rightarrow_i^*$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_e$, that is, for some formula B

$$\mathcal{D}_{\text{seq}} = \frac{\frac{\vdots \mathcal{D}'}{\Gamma \vdash B \Rightarrow A} \quad \frac{\vdots \mathcal{D}''}{\Gamma \vdash B}}{\Gamma \vdash A} \Rightarrow_e$$

where $\mathcal{D}'$ and $\mathcal{D}''$ are the smallest derivation in ND_{seq} that prove the sequents $\Gamma \vdash B \Rightarrow A$ and $\Gamma \vdash B$, respectively, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$ and $\mathcal{D}''$, respectively, there are derivations $\mathcal{D}'_0$ and $\mathcal{D}''_0$ in ND that prove $\Gamma \vdash B \Rightarrow A$ and $\Gamma \vdash B$. Then, the derivation $\mathcal{D}_0$ below prove that $\Gamma \vdash A$ in ND .

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}'}{B \Rightarrow A} \quad \frac{\vdots \mathcal{D}''}{B}}{A} \Rightarrow_e$$

□

Exercise 6

For any formula B , prove that if $\Gamma \vdash A$ is derivable in ND_{seq} , then so is $\Gamma, B \vdash A$.

Solution to Exercise 6

By induction on the number of rules of the smallest derivation $\mathcal{D}$ in ND_{seq} proving the sequent $\Gamma \vdash A$. Cases:

- The last rule of $\mathcal{D}$ is ax , that is,

$$\mathcal{D} = \frac{}{\Gamma', A \vdash A} \text{ax}$$

where $\Gamma = \Gamma', A$ for some finite multiset Γ' . Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq} .

$$\frac{}{\Gamma', B, A \vdash A} \text{ax}$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_i$, that is, $A = D \Rightarrow C$ and

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}'}{\Gamma, D \vdash C}}{\Gamma \vdash D \Rightarrow C} \Rightarrow_i$$

where $\mathcal{D}'$ is the smallest derivation in ND_{seq} of the sequent $\Gamma, D \vdash C$, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$, there is a derivation $\mathcal{D}_0$ in ND_{seq} that proves the sequent $\Gamma, B, D \vdash C$. Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq} .

$$\frac{\begin{array}{c} \vdots \mathcal{D}_0 \\ \Gamma, B, D \vdash C \end{array}}{\Gamma, B \vdash D \Rightarrow C} \Rightarrow_i$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_e$, that is, for some formula C

$$\mathcal{D}_{\text{seq}} = \frac{\begin{array}{c} \vdots \mathcal{D}' \\ \Gamma \vdash C \Rightarrow A \end{array} \quad \begin{array}{c} \vdots \mathcal{D}'' \\ \Gamma \vdash C \end{array}}{\Gamma \vdash A} \Rightarrow_e$$

where $\mathcal{D}'$ and $\mathcal{D}''$ are the smallest derivations in ND_{seq} that prove the sequents $\Gamma \vdash C \Rightarrow A$ and $\Gamma \vdash C$, respectively, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$ and $\mathcal{D}''$, respectively, there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ in ND_{seq} that prove the sequents $\Gamma, B \vdash C \Rightarrow A$ and $\Gamma \vdash C$. Then, the derivation below prove the sequent $\Gamma, B \vdash A$ in ND_{seq} .

$$\frac{\begin{array}{c} \vdots \mathcal{D}_1 \\ \Gamma, B \vdash C \Rightarrow A \end{array} \quad \begin{array}{c} \vdots \mathcal{D}_2 \\ \Gamma, B \vdash C \end{array}}{\Gamma, B \vdash A} \Rightarrow_e$$

Exercise 7

For any formula B , prove that if $\Gamma, B, B \vdash A$ is derivable in ND_{seq} then so is $\Gamma, B \vdash A$.

Solution to Exercise 7

By induction on the number of rules of the smallest derivation $\mathcal{D}$ in ND_{seq} proving the sequent $\Gamma, B, B \vdash A$. Cases:

- The last rule of $\mathcal{D}$ is ax , that is,

$$\mathcal{D} = \frac{}{\Gamma', B, B, A \vdash A} \text{ax}$$

where $\Gamma = \Gamma', A$ for some finite multiset Γ' . Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq} .

$$\frac{}{\Gamma', B, A \vdash A} \text{ax}$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_i$, that is, $A = D \Rightarrow C$ and

$$\mathcal{D} = \frac{\begin{array}{c} \vdots \mathcal{D}' \\ \Gamma, B, D \vdash C \end{array}}{\Gamma, B, B \vdash D \Rightarrow C} \Rightarrow_i$$

where $\mathcal{D}'$ is the smallest derivation in ND_{seq} of the sequent $\Gamma, B, B, D \vdash C$, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$, there is a derivation $\mathcal{D}_0$ in ND_{seq} that proves the sequent $\Gamma, B, D \vdash C$. Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq} .

$$\frac{\begin{array}{c} \vdots \mathcal{D}_0 \\ \Gamma, B, D \vdash C \end{array}}{\Gamma, B \vdash D \Rightarrow C} \Rightarrow_i$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_e$, that is, for some formula C

$$\mathcal{D}_{\text{seq}} = \frac{\frac{\vdots \mathcal{D}'}{\Gamma, B, B \vdash C \Rightarrow A} \quad \frac{\vdots \mathcal{D}''}{\Gamma, B, B \vdash C}}{\Gamma, B, B \vdash A} \Rightarrow_e$$

where $\mathcal{D}'$ and $\mathcal{D}''$ are the smallest derivations in ND_{seq} that prove the sequents $\Gamma, B, B \vdash C \Rightarrow A$ and $\Gamma, B, B \vdash C$, respectively, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$ and $\mathcal{D}''$, respectively, there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ in ND_{seq} that prove the sequents $\Gamma, B \vdash C \Rightarrow A$ and $\Gamma \vdash C$. Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq} .

$$\frac{\frac{\vdots \mathcal{D}_1}{\Gamma, B \vdash C \Rightarrow A} \quad \frac{\vdots \mathcal{D}_2}{\Gamma, B \vdash C}}{\Gamma, B \vdash A} \Rightarrow_e$$