

The λ -calculus, typed and untyped

Lecture 2: The simply typed λ -calculus and the Curry-Howard correspondence

Giulio Guerrieri

✉ giulio.guerrieri@uibk.ac.at 🌐 <https://cl-teaching.uibk.ac.at/26w/1c>

Institut für Informatik, Universität Innsbruck

Innsbruck, Austria – 23 October 2026

Outline

- 1 Natural deduction for minimal logic, revisited
- 2 The Curry-Howard correspondence, informally
- 3 The simply typed λ -calculus
- 4 Conclusion, exercises and bibliography

Outline

- 1 Natural deduction for minimal logic, revisited
- 2 The Curry-Howard correspondence, informally
- 3 The simply typed λ -calculus
- 4 Conclusion, exercises and bibliography

ND revisited

Def. We fix an infinite set of **labels** ranged over by $x, y, z, \dots$. A **labeled formula** is a pair (x, A) , also denoted by $x : A$ or A^x , where A is a formula and x is a label. A set Γ of labeled formulas is **coherent** if $A = B$ whenever $A^x, B^x \in \Gamma$.

Def. A **derivation** $\mathcal{D}$ in ND with **hypotheses** $h(\mathcal{D})$ and **conclusion** $c(\mathcal{D})$ is:

- either A^x for any formula A and any label x , with $h(\mathcal{D}) = \{A^x\}$ and $c(\mathcal{D}) = A$;
- or it is obtained from derivations $\mathcal{D}', \mathcal{D}_1, \mathcal{D}_2$ by applying one of the inference rules

$\Rightarrow$ elimination

$$\frac{\begin{array}{c} \vdots \\ \mathcal{D}_1 \\ A \Rightarrow B \end{array} \quad \begin{array}{c} \vdots \\ \mathcal{D}_2 \\ A \end{array}}{B} \Rightarrow_e \quad \text{provided that } h(\mathcal{D}_1) \cup h(\mathcal{D}_2) \text{ is coherent}$$

with $h(\mathcal{D}) = h(\mathcal{D}_1) \cup h(\mathcal{D}_2)$
and $c(\mathcal{D}) = B$

$\Rightarrow$ introduction

$$\frac{\begin{array}{c} [A^x] \\ \vdots \\ \mathcal{D}' \\ B \end{array}}{A \Rightarrow B} \Rightarrow_i^x \quad \text{provided that all occurrences of } A^x \text{ in } \mathcal{D}' \text{ are discarded by } \Rightarrow_i^x$$

with $h(\mathcal{D}) = h(\mathcal{D}') \setminus \{A^x\}$
and $c(\mathcal{D}) = A \Rightarrow B$

Idea. The label x determines how many occurrences of A are discharged by $\Rightarrow_i^x$.

ND revisited

Def. We fix an infinite set of **labels** ranged over by $x, y, z, \dots$. A **labeled formula** is a pair (x, A) , also denoted by $x : A$ or A^x , where A is a formula and x is a label. A set Γ of labeled formulas is **coherent** if $A = B$ whenever $A^x, B^x \in \Gamma$.

Def. A **derivation** $\mathcal{D}$ in ND with **hypotheses** $h(\mathcal{D})$ and **conclusion** $c(\mathcal{D})$ is:

- either A^x for any formula A and any label x , with $h(\mathcal{D}) = \{A^x\}$ and $c(\mathcal{D}) = A$;
- or it is obtained from derivations $\mathcal{D}', \mathcal{D}_1, \mathcal{D}_2$ by applying one of the inference rules

$\Rightarrow$ elimination

$$\frac{\begin{array}{c} \vdots \\ \mathcal{D}_1 \\ \vdots \\ A \Rightarrow B \end{array} \quad \begin{array}{c} \vdots \\ \mathcal{D}_2 \\ \vdots \\ A \end{array} \quad \begin{array}{l} \text{provided that} \\ h(\mathcal{D}_1) \cup h(\mathcal{D}_2) \\ \text{is coherent} \end{array}}{B} \Rightarrow_e$$

with $h(\mathcal{D}) = h(\mathcal{D}_1) \cup h(\mathcal{D}_2)$
and $c(\mathcal{D}) = B$

$\Rightarrow$ introduction

$$\frac{\begin{array}{c} [A^x] \\ \vdots \\ \mathcal{D}' \\ \vdots \\ B \end{array} \quad \begin{array}{l} \text{provided that all} \\ \text{occurrences of } A^x \text{ in } \mathcal{D}' \\ \text{are discarded by } \Rightarrow_i^x \end{array}}{A \Rightarrow B} \Rightarrow_i^x$$

with $h(\mathcal{D}) = h(\mathcal{D}') \setminus \{A^x\}$
and $c(\mathcal{D}) = A \Rightarrow B$

Idea. The label x determines how many occurrences of A are discharged by $\Rightarrow_i^x$.

ND revisited

Def. We fix an infinite set of **labels** ranged over by $x, y, z, \dots$. A **labeled formula** is a pair (x, A) , also denoted by $x : A$ or A^x , where A is a formula and x is a label. A set Γ of labeled formulas is **coherent** if $A = B$ whenever $A^x, B^x \in \Gamma$.

Def. A **derivation** $\mathcal{D}$ in ND with **hypotheses** $h(\mathcal{D})$ and **conclusion** $c(\mathcal{D})$ is:

- either A^x for any formula A and any label x , with $h(\mathcal{D}) = \{A^x\}$ and $c(\mathcal{D}) = A$;
- or it is obtained from derivations $\mathcal{D}', \mathcal{D}_1, \mathcal{D}_2$ by applying one of the inference rules

$\Rightarrow$ **elimination**

$$\frac{\begin{array}{c} \vdots \\ \mathcal{D}_1 \\ \hline A \Rightarrow B \end{array} \quad \begin{array}{c} \vdots \\ \mathcal{D}_2 \\ \hline A \end{array} \quad \begin{array}{l} \text{provided that} \\ h(\mathcal{D}_1) \cup h(\mathcal{D}_2) \\ \text{is coherent} \end{array}}{B} \Rightarrow_e$$

with $h(\mathcal{D}) = h(\mathcal{D}_1) \cup h(\mathcal{D}_2)$
and $c(\mathcal{D}) = B$

$\Rightarrow$ **introduction**

$$\frac{\begin{array}{c} [A^x] \\ \vdots \\ \mathcal{D}' \\ \hline B \end{array} \quad \begin{array}{l} \text{provided that all} \\ \text{occurrences of } A^x \text{ in } \mathcal{D}' \\ \text{are discarded by } \Rightarrow_i^x \end{array}}{A \Rightarrow B} \Rightarrow_i^x$$

with $h(\mathcal{D}) = h(\mathcal{D}') \setminus \{A^x\}$
and $c(\mathcal{D}) = A \Rightarrow B$

Idea. The label x determines how many occurrences of A are discharged by $\Rightarrow_i^x$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- 1 Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.
- 2 Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.
- 3 Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

A^x

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{[A^x]}{B \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{[A^x]}{B \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{[A^x]}{B \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{A^x}{A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{[A^x]}{B \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{A^x}{A \Rightarrow A} \Rightarrow_i^y \neq \frac{[A^y]}{A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow_i^x A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow_i^x A} \Rightarrow_i^x \quad \frac{\frac{[A^x]}{B \Rightarrow_i^y A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{A^x}{A \Rightarrow_i^y A} \Rightarrow_i^y \neq \frac{[A^y]}{A \Rightarrow_i^y A} \Rightarrow_i^y \quad \frac{\frac{[A^x]}{A \Rightarrow_i^y A} \Rightarrow_i^y}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^x$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

Notation. $A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ (or $A_1^{x_1}, \dots, A_n^{x_n} \vdash_{\text{ND}} B$) means that there is a derivation in ND with conclusion B and hypotheses *among* the labeled formulas $A_1^{x_1}, \dots, A_n^{x_n}$.

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{[A^x]}{B \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{A^x}{A \Rightarrow A} \Rightarrow_i^y \neq \frac{[A^y]}{A \Rightarrow A} \Rightarrow_i^y \quad \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \neq \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c}
 A^x \\
 \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x
 \end{array}
 \qquad
 \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x
 \qquad
 \frac{\frac{[A^x]}{B \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{A^x}{A \Rightarrow A} \Rightarrow_i^y \neq \frac{[A^y]}{A \Rightarrow A} \Rightarrow_i^y
 \qquad
 \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \neq \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND, revisited

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{[A^x]}{B \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{A^x}{A \Rightarrow A} \Rightarrow_i^y \neq \frac{[A^y]}{A \Rightarrow A} \Rightarrow_i^y \quad \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \neq \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

$$\frac{\frac{[(A \Rightarrow B)^x] \quad [A^z]}{B} \Rightarrow_e}{\frac{C}{A \Rightarrow C} \Rightarrow_i^z} \Rightarrow_e \quad \frac{\frac{C}{A \Rightarrow C} \Rightarrow_i^z}{(B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^y \quad \frac{(B \Rightarrow C) \Rightarrow A \Rightarrow C}{(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^x$$

Examples of derivations in ND, revisited

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{[A^x]}{B \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{A^x}{A \Rightarrow A} \Rightarrow_i^y \neq \frac{[A^y]}{A \Rightarrow A} \Rightarrow_i^y \quad \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \neq \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

$$\frac{\frac{\frac{[(A \Rightarrow B)^x]}{(B \Rightarrow C)^y} \Rightarrow_e \quad \frac{[A^z]}{B} \Rightarrow_e}{C} \Rightarrow_e}{\frac{A \Rightarrow C \Rightarrow_i^z}{(B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^y} \Rightarrow_i^x \quad \frac{\frac{\frac{[(A \Rightarrow B \Rightarrow C)^x]}{B \Rightarrow C} \Rightarrow_e \quad \frac{[A^z]}{B} \Rightarrow_e}{C} \Rightarrow_e}{\frac{A \Rightarrow C \Rightarrow_i^z}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^y} \Rightarrow_i^x$$

Examples of derivations in ND, revisited

- ❶ Prove that $A^x \vdash A$, and $\vdash A \Rightarrow A$, and $B^y \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$A^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{[A^x]}{B \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $A^x \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{A^x}{A \Rightarrow A} \Rightarrow_i^y \neq \frac{[A^y]}{A \Rightarrow A} \Rightarrow_i^y \quad \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^y}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \neq \frac{\frac{[A^x]}{A \Rightarrow A} \Rightarrow_i^x}{A \Rightarrow A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

$$\frac{\frac{\frac{[(A \Rightarrow B)^x]}{(B \Rightarrow C)^y} \quad \frac{[A^z]}{B} \Rightarrow_e}{C} \Rightarrow_e}{\frac{A \Rightarrow C \Rightarrow_i^z}{(B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^y} \Rightarrow_i^x \quad \frac{\frac{\frac{[(A \Rightarrow B \Rightarrow C)^x]}{B \Rightarrow C} \quad \frac{[A^z]}{B} \Rightarrow_e}{C} \Rightarrow_e}{\frac{A \Rightarrow C \Rightarrow_i^z}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^y} \Rightarrow_i^x$$

An alternative presentation of ND via sequents: ND_{seq} , revisited

Def. An **environment** is a partial function with finite domain from labels to formulas. An environment Γ is denoted by $x_1 : A_1, \dots, x_n : A_n$ if $\text{dom}(\Gamma) = \{x_1, \dots, x_n\}$ and $\Gamma(x_i) = A_i$. A **sequent** is a pair $\Gamma \vdash A$ of an environment Γ and a formula A .

Def. A **derivation** in ND_{seq} is a finite tree built up from the inference rules below.

$$\frac{}{\Gamma, x : A \vdash A} \text{ax}^x \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow_e \qquad \frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow_i^x$$

Notations. $\mathcal{D} \triangleright_{\text{ND}_{\text{seq}}} \Gamma \vdash A$ means that $\mathcal{D}$ is a derivation in ND_{seq} with conclusion $\Gamma \vdash A$. $\Gamma \vdash_{\text{ND}_{\text{seq}}} A$ means that $\Gamma \vdash A$ is **derivable** in ND_{seq} i.e. there is a derivation $\mathcal{D} \triangleright_{\text{ND}_{\text{seq}}} \Gamma \vdash A$.

Idea. In ND_{seq} a derivation makes explicit:

- 1 the current hypotheses at each inference step;
- 2 if two occurrences of the same formula should be treated separately (different labels).

Rmk. $x : X, y : X$ is an environment, while $x : X, x : X \Rightarrow X$ is not an environment!

An alternative presentation of ND via sequents: ND_{seq} , revisited

Def. An **environment** is a partial function with finite domain from labels to formulas. An environment Γ is denoted by $x_1 : A_1, \dots, x_n : A_n$ if $\text{dom}(\Gamma) = \{x_1, \dots, x_n\}$ and $\Gamma(x_i) = A_i$. A **sequent** is a pair $\Gamma \vdash A$ of an environment Γ and a formula A .

Def. A **derivation** in ND_{seq} is a finite tree built up from the inference rules below.

$$\frac{}{\Gamma, x : A \vdash A} \text{ax}^x \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow_e \qquad \frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow_i^x$$

Notations. $\mathcal{D} \triangleright_{\text{ND}_{\text{seq}}} \Gamma \vdash A$ means that $\mathcal{D}$ is a derivation in ND_{seq} with conclusion $\Gamma \vdash A$. $\Gamma \vdash_{\text{ND}_{\text{seq}}} A$ means that $\Gamma \vdash A$ is **derivable** in ND_{seq} i.e. there is a derivation $\mathcal{D} \triangleright_{\text{ND}_{\text{seq}}} \Gamma \vdash A$.

Idea. In ND_{seq} a derivation makes explicit:

- 1 the current hypotheses at each inference step;
- 2 if two occurrences of the same formula should be treated separately (different labels).

Rmk. $x : X, y : X$ is an environment, while $x : X, x : X \Rightarrow X$ is not an environment!

An alternative presentation of ND via sequents: ND_{seq} , revisited

Def. An **environment** is a partial function with finite domain from labels to formulas. An environment Γ is denoted by $x_1 : A_1, \dots, x_n : A_n$ if $\text{dom}(\Gamma) = \{x_1, \dots, x_n\}$ and $\Gamma(x_i) = A_i$. A **sequent** is a pair $\Gamma \vdash A$ of an environment Γ and a formula A .

Def. A **derivation** in ND_{seq} is a finite tree built up from the inference rules below.

$$\frac{}{\Gamma, x : A \vdash A} \text{ax}^x \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow_e \qquad \frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow_i^x$$

Notations. $\mathcal{D} \triangleright_{\text{ND}_{\text{seq}}} \Gamma \vdash A$ means that $\mathcal{D}$ is a derivation in ND_{seq} with conclusion $\Gamma \vdash A$. $\Gamma \vdash_{\text{ND}_{\text{seq}}} A$ means that $\Gamma \vdash A$ is **derivable** in ND_{seq} i.e. there is a derivation $\mathcal{D} \triangleright_{\text{ND}_{\text{seq}}} \Gamma \vdash A$.

Idea. In ND_{seq} a derivation makes explicit:

- 1 the current hypotheses at each inference step;
- 2 if two occurrences of the same formula should be treated separately (different labels).

Rmk. $x : X, y : X$ is an environment, while $x : X, x : X \Rightarrow X$ is not an environment!

Examples of derivations in ND_{seq} , revisited

- 1 Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.
- 2 Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.
- 3 Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\frac{}{x : A \vdash A} \text{ax}^x$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\frac{}{x : A \vdash A} \text{ax}^x \qquad \frac{\frac{}{x : A \vdash A} \text{ax}^x}{\vdash A \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\frac{}{x : A \vdash A} \text{ax}^x \qquad \frac{}{x : A \vdash A} \text{ax}^x \qquad \frac{}{x : A, y : B \vdash A} \text{ax}^x$$
$$\frac{}{\vdash A \Rightarrow A} \Rightarrow_i^x \qquad \frac{}{\vdash A \Rightarrow A} \Rightarrow_i^x \qquad \frac{}{y : B \vdash A \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c}
 \frac{}{x : A \vdash A} \text{ax}^x
 \end{array}
 \qquad
 \frac{}{x : A \vdash A} \text{ax}^x
 \qquad
 \frac{}{x : A, y : B \vdash A} \text{ax}^x
 \qquad
 \frac{}{x : A, y : B \vdash A} \text{ax}^x$$

$$\frac{}{\vdash A \Rightarrow A} \Rightarrow_i^x
 \qquad
 \frac{}{y : B \vdash A \Rightarrow A} \Rightarrow_i^x
 \qquad
 \frac{}{x : A \vdash B \Rightarrow A} \Rightarrow_i^y
 \qquad
 \frac{}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c}
 \frac{}{x : A \vdash A} \text{ax}^x
 \end{array}
 \qquad
 \frac{}{x : A \vdash A} \text{ax}^x
 \qquad
 \frac{}{x : A, y : B \vdash A} \text{ax}^x
 \qquad
 \frac{}{x : A, y : B \vdash A} \text{ax}^x$$

$$\frac{}{\vdash A \Rightarrow A} \Rightarrow_i^x
 \qquad
 \frac{}{y : B \vdash A \Rightarrow A} \Rightarrow_i^x
 \qquad
 \frac{}{x : A \vdash B \Rightarrow A} \Rightarrow_i^y
 \qquad
 \frac{}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c}
 \frac{}{x : A \vdash A} \text{ax}^x
 \end{array}
 \qquad
 \frac{}{x : A \vdash A} \text{ax}^x
 \quad
 \frac{}{\vdash A \Rightarrow A} \Rightarrow_i^x
 \qquad
 \frac{}{x : A, y : B \vdash A} \text{ax}^x
 \quad
 \frac{}{y : B \vdash A \Rightarrow A} \Rightarrow_i^x
 \qquad
 \frac{}{x : A, y : B \vdash A} \text{ax}^x
 \quad
 \frac{}{x : A \vdash B \Rightarrow A} \Rightarrow_i^y
 \quad
 \frac{}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{}{y : A, x : A \vdash A} \text{ax}^x
 \quad
 \frac{}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c}
 \frac{}{x : A \vdash A} \text{ax}^x \\
 \frac{}{x : A \vdash A} \text{ax}^x \\
 \frac{}{\vdash A \Rightarrow A} \Rightarrow_i^x \\
 \frac{}{x : A, y : B \vdash A} \text{ax}^x \\
 \frac{}{y : B \vdash A \Rightarrow A} \Rightarrow_i^x \\
 \frac{}{x : A, y : B \vdash A} \text{ax}^x \\
 \frac{}{x : A \vdash B \Rightarrow A} \Rightarrow_i^y \\
 \frac{}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_i^x
 \end{array}$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{}{y : A, x : A \vdash A} \text{ax}^x \neq \frac{}{y : A, x : A \vdash A} \text{ax}^y \\
 \frac{}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \neq \frac{}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c} \frac{}{x : A \vdash A} \text{ax}^x \qquad \frac{}{x : A \vdash A} \text{ax}^x \qquad \frac{}{x : A, y : B \vdash A} \text{ax}^x \qquad \frac{}{x : A, y : B \vdash A} \text{ax}^x \\ \vdash A \Rightarrow A \Rightarrow_i^x \qquad \vdash A \Rightarrow A \Rightarrow_i^x \qquad \vdash A \Rightarrow A \Rightarrow_i^x \qquad \frac{}{x : A \vdash B \Rightarrow A} \Rightarrow_i^y \\ \vdash A \Rightarrow B \Rightarrow A \Rightarrow_i^x \end{array}$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\begin{array}{c} \frac{}{y : A, x : A \vdash A} \text{ax}^x \qquad \frac{}{y : A, x : A \vdash A} \text{ax}^y \qquad \frac{}{y : A, x : A \vdash A} \text{ax}^x \\ \vdash A \Rightarrow A \Rightarrow_i^y \neq \vdash A \Rightarrow A \Rightarrow_i^y \qquad \frac{}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \\ \vdash A \Rightarrow A \Rightarrow A \Rightarrow_i^x \end{array}$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c}
 \frac{}{x : A \vdash A} ax^x \\
 \frac{}{x : A \vdash A} ax^x \quad \frac{}{\vdash A \Rightarrow A} \Rightarrow_i^x \\
 \frac{}{x : A, y : B \vdash A} ax^x \quad \frac{}{y : B \vdash A \Rightarrow A} \Rightarrow_i^x \\
 \frac{}{x : A, y : B \vdash A} ax^x \quad \frac{}{x : A \vdash B \Rightarrow A} \Rightarrow_i^y \quad \frac{}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_i^x
 \end{array}$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\begin{array}{c}
 \frac{}{y : A, x : A \vdash A} ax^x \quad \frac{}{y : A, x : A \vdash A} ax^y \quad \frac{}{y : A, x : A \vdash A} ax^x \quad \frac{}{y : A, x : A \vdash A} ax^x \\
 \frac{}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \neq \frac{}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \quad \frac{}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \neq \frac{}{y : A \vdash A \Rightarrow A} \Rightarrow_i^x \quad \frac{}{\vdash A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \neq \frac{}{\vdash A \Rightarrow A \Rightarrow A} \Rightarrow_i^y
 \end{array}$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\frac{}{x : A \vdash A} \text{ax}^x \quad \frac{\frac{}{x : A \vdash A} \text{ax}^x}{\vdash A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{}{x : A, y : B \vdash A} \text{ax}^x}{y : B \vdash A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{\frac{}{x : A, y : B \vdash A} \text{ax}^x}{x : A \vdash B \Rightarrow A} \Rightarrow_i^y}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{\frac{}{y : A, x : A \vdash A} \text{ax}^x}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \neq \frac{\frac{}{y : A, x : A \vdash A} \text{ax}^y}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \quad \frac{\frac{\frac{}{y : A, x : A \vdash A} \text{ax}^x}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y}{\vdash A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \neq \frac{\frac{\frac{}{y : A, x : A \vdash A} \text{ax}^x}{y : A \vdash A \Rightarrow A} \Rightarrow_i^x}{\vdash A \Rightarrow A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c} \overline{\quad}^{ax^x} \\ x : A \vdash A \end{array} \quad \frac{\overline{x : A \vdash A}^{ax^x}}{\vdash A \Rightarrow A} \Rightarrow_i^x \quad \frac{\overline{x : A, y : B \vdash A}^{ax^x}}{y : B \vdash A \Rightarrow A} \Rightarrow_i^x \quad \frac{\overline{x : A, y : B \vdash A}^{ax^x}}{x : A \vdash B \Rightarrow A} \Rightarrow_i^y \quad \frac{x : A \vdash B \Rightarrow A}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{\overline{y : A, x : A \vdash A}^{ax^x}}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \neq \frac{\overline{y : A, x : A \vdash A}^{ax^y}}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \quad \frac{\overline{y : A, x : A \vdash A}^{ax^x}}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \neq \frac{\overline{y : A, x : A \vdash A}^{ax^x}}{y : A \vdash A \Rightarrow A} \Rightarrow_i^x \quad \frac{x : A \vdash A \Rightarrow A}{\vdash A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \neq \frac{y : A \vdash A \Rightarrow A}{\vdash A \Rightarrow A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

$$\begin{array}{c} \overline{\Gamma \vdash B \Rightarrow C}^{ax^y} \quad \frac{\overline{\Gamma \vdash A \Rightarrow B}^{ax^x} \quad \overline{\Gamma \vdash A}^{ax^z}}{\Gamma \vdash B} \Rightarrow_e \\ \frac{\Gamma \vdash B \Rightarrow C}{z : A, x : A \Rightarrow B, y : B \Rightarrow C \vdash C} \Rightarrow_e \\ \frac{z : A, x : A \Rightarrow B, y : B \Rightarrow C \vdash C}{x : A \Rightarrow B, y : B \Rightarrow C \vdash A \Rightarrow C} \Rightarrow_i^z \\ \frac{x : A \Rightarrow B, y : B \Rightarrow C \vdash A \Rightarrow C}{x : A \Rightarrow B \vdash (B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^y \\ \frac{x : A \Rightarrow B \vdash (B \Rightarrow C) \Rightarrow A \Rightarrow C}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^x \end{array}$$

where $\Gamma = z : A, x : A \Rightarrow B, y : B \Rightarrow C$

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c} \overline{\quad}^{ax^x} \\ x : A \vdash A \end{array} \quad \frac{\overline{x : A \vdash A}^{ax^x}}{\vdash A \Rightarrow A} \Rightarrow_i^x \quad \frac{\overline{x : A, y : B \vdash A}^{ax^x}}{y : B \vdash A \Rightarrow A} \Rightarrow_i^x \quad \frac{\overline{x : A, y : B \vdash A}^{ax^x}}{x : A \vdash B \Rightarrow A} \Rightarrow_i^y \quad \frac{\quad}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_i^x$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\frac{\overline{y : A, x : A \vdash A}^{ax^x}}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \neq \frac{\overline{y : A, x : A \vdash A}^{ax^y}}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \quad \frac{\overline{y : A, x : A \vdash A}^{ax^x}}{x : A \vdash A \Rightarrow A} \Rightarrow_i^y \neq \frac{\overline{y : A, x : A \vdash A}^{ax^x}}{y : A \vdash A \Rightarrow A} \Rightarrow_i^x \quad \frac{\quad}{\vdash A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \neq \frac{\quad}{\vdash A \Rightarrow A \Rightarrow A} \Rightarrow_i^y$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

$$\begin{array}{c} \overline{\quad}^{ax^y} \quad \frac{\overline{\Gamma \vdash A \Rightarrow B}^{ax^x} \quad \overline{\Gamma \vdash A}^{ax^z}}{\Gamma \vdash B \Rightarrow C} \Rightarrow_e \quad \frac{\quad}{\Gamma \vdash B} \Rightarrow_e \\ \frac{z : A, x : A \Rightarrow B, y : B \Rightarrow C \vdash C}{x : A \Rightarrow B, y : B \Rightarrow C \vdash A \Rightarrow C} \Rightarrow_e \quad \frac{\quad}{x : A \Rightarrow B \vdash (B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^z \\ \frac{\quad}{x : A \Rightarrow B \vdash (B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^y \\ \frac{\quad}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^x \end{array} \quad \begin{array}{c} \overline{\Gamma \vdash A \Rightarrow B \Rightarrow C}^{ax^x} \quad \overline{\Gamma \vdash A}^{ax^z} \quad \frac{\overline{\Gamma \vdash A \Rightarrow B}^{ax^y} \quad \overline{\Gamma \vdash A}^{ax^z}}{\Gamma \vdash B} \Rightarrow_e \\ \frac{\quad}{\Gamma \vdash B \Rightarrow C} \Rightarrow_e \quad \frac{\quad}{\Gamma \vdash B} \Rightarrow_e \\ \frac{z : A, y : A \Rightarrow B, x : A \Rightarrow B \Rightarrow C \vdash C}{y : A \Rightarrow B, x : A \Rightarrow B \Rightarrow C \vdash A \Rightarrow C} \Rightarrow_e \quad \frac{\quad}{y : A \Rightarrow B, x : A \Rightarrow B \Rightarrow C \vdash A \Rightarrow C} \Rightarrow_i^z \\ \frac{\quad}{x : A \Rightarrow B \Rightarrow C \vdash (A \Rightarrow B) \Rightarrow A \Rightarrow C} \Rightarrow_i^y \\ \frac{\quad}{\vdash (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C} \Rightarrow_i^x \end{array}$$

where $\Gamma = z : A, x : A \Rightarrow B, y : B \Rightarrow C$

where $\Gamma = z : A, y : A \Rightarrow B, x : A \Rightarrow B \Rightarrow C$

Examples of derivations in ND_{seq} , revisited

- ❶ Prove $x : A \vdash A$, and $\vdash A \Rightarrow A$, and $y : B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

$$\begin{array}{c} \overline{x : A \vdash A}^{ax^x} \\ \overline{x : A \vdash A}^{ax^x} \Rightarrow_i^x \vdash A \Rightarrow A \\ \overline{x : A, y : B \vdash A}^{ax^x} \Rightarrow_i^x y : B \vdash A \Rightarrow A \\ \overline{x : A, y : B \vdash A}^{ax^x} \Rightarrow_i^y x : A \vdash B \Rightarrow A \Rightarrow_i^x \vdash A \Rightarrow B \Rightarrow A \end{array}$$

- ❷ Give two distinct derivations of $x : A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\begin{array}{c} \overline{y : A, x : A \vdash A}^{ax^x} \Rightarrow_i^y x : A \vdash A \Rightarrow A \\ \overline{y : A, x : A \vdash A}^{ax^y} \Rightarrow_i^y x : A \vdash A \Rightarrow A \\ \overline{y : A, x : A \vdash A}^{ax^x} \Rightarrow_i^y x : A \vdash A \Rightarrow A \Rightarrow_i^x \vdash A \Rightarrow A \Rightarrow A \\ \overline{y : A, x : A \vdash A}^{ax^x} \Rightarrow_i^y x : A \vdash A \Rightarrow A \Rightarrow_i^x \vdash A \Rightarrow A \Rightarrow A \end{array}$$

- ❸ Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

$$\begin{array}{c} \overline{\Gamma \vdash A \Rightarrow B}^{ax^x} \overline{\Gamma \vdash A}^{ax^z} \Rightarrow_e \overline{\Gamma \vdash B \Rightarrow C}^{ax^y} \Rightarrow_e \overline{\Gamma \vdash B} \Rightarrow_e \overline{\Gamma \vdash A \Rightarrow B \Rightarrow C}^{ax^x} \overline{\Gamma \vdash A}^{ax^z} \Rightarrow_e \overline{\Gamma \vdash A \Rightarrow B}^{ax^y} \overline{\Gamma \vdash A}^{ax^z} \Rightarrow_e \\ \overline{z : A, x : A \Rightarrow B, y : B \Rightarrow C \vdash C} \Rightarrow_e \overline{z : A, y : A \Rightarrow B, x : A \Rightarrow B \Rightarrow C \vdash C} \Rightarrow_e \overline{y : A \Rightarrow B, x : A \Rightarrow B \Rightarrow C \vdash A \Rightarrow C} \Rightarrow_i^z \overline{x : A \Rightarrow B \vdash (B \Rightarrow C) \Rightarrow A \Rightarrow C} \Rightarrow_i^y \overline{x : A \Rightarrow B \Rightarrow C \vdash (A \Rightarrow B) \Rightarrow A \Rightarrow C} \Rightarrow_i^y \overline{\vdash (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C} \Rightarrow_i^x \end{array}$$

where $\Gamma = z : A, x : A \Rightarrow B, y : B \Rightarrow C$

where $\Gamma = z : A, y : A \Rightarrow B, x : A \Rightarrow B \Rightarrow C$

Outline

- 1 Natural deduction for minimal logic, revisited
- 2 The Curry-Howard correspondence, informally
- 3 The simply typed λ -calculus
- 4 Conclusion, exercises and bibliography

A computational interpretation of ND

Idea. A derivation $\mathcal{D} \triangleright_{\text{ND}} A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ can be seen as a **function** t

$$\begin{array}{c} A_1^{x_1} \quad \vdots \quad A_n^{x_n} \\ \text{---} \\ \triangle \text{---} \\ \mathcal{D} \\ \text{---} \\ B \end{array} \qquad t = (x_1, \dots, x_n) \mapsto t(x_1, \dots, x_n)$$

that associates with derivations $\mathcal{D}_1 \triangleright_{\text{ND}} \vdash A_1, \dots, \mathcal{D}_n \triangleright_{\text{ND}} \vdash A_n$,

$$\begin{array}{c} \triangle \text{---} \\ \mathcal{D}_1 \\ \text{---} \\ A_1 \end{array} \quad \vdots \quad \begin{array}{c} \triangle \text{---} \\ \mathcal{D}_n \\ \text{---} \\ A_n \end{array}$$

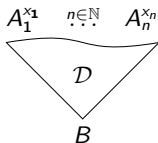
a derivation $t(\mathcal{D}_1, \dots, \mathcal{D}_n) \triangleright_{\text{ND}} \vdash B$ obtained by composing $\mathcal{D}$ with $\mathcal{D}_1, \dots, \mathcal{D}_n$.

$$\begin{array}{c} \triangle \text{---} \quad \triangle \text{---} \\ \mathcal{D}_1 \quad \mathcal{D}_n \\ \text{---} \quad \text{---} \\ A_1 \quad \vdots \quad A_n \\ \text{---} \\ \triangle \text{---} \\ \mathcal{D} \\ \text{---} \\ B \end{array}$$

↪ Let us see how, by induction on $\mathcal{D}$.

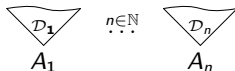
A computational interpretation of ND

Idea. A derivation $\mathcal{D} \triangleright_{\text{ND}} A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ can be seen as a function t

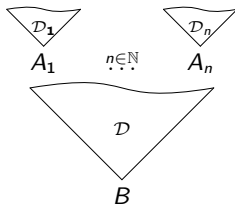


$$t = (x_1, \dots, x_n) \mapsto t(x_1, \dots, x_n)$$

that associates with derivations $\mathcal{D}_1 \triangleright_{\text{ND}} \vdash A_1, \dots, \mathcal{D}_n \triangleright_{\text{ND}} \vdash A_n$,



a derivation $t(\mathcal{D}_1, \dots, \mathcal{D}_n) \triangleright_{\text{ND}} \vdash B$ obtained by composing $\mathcal{D}$ with $\mathcal{D}_1, \dots, \mathcal{D}_n$.



Let us see how, by induction on $\mathcal{D}$.

A computational interpretation of ND: variable

Let the derivation $\mathcal{D} \triangleright_{\text{ND}} A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ consist of a single hypothesis $A_i^{x_i}$, that is,

$$\mathcal{D} = A_i^{x_i} \quad (\text{with } B = A_i)$$

The derivation $\mathcal{D}$ can then be seen a function

$$t = (x_1, \dots, x_n) \mapsto x_i \quad (\approx \text{projection})$$

represented by the **variable** x_i . Indeed, given the arguments $\mathcal{D}_1 \triangleright_{\text{ND}} \vdash A_1, \dots, \mathcal{D}_n \triangleright_{\text{ND}} \vdash A_n$,

$$\begin{array}{c} \triangle \mathcal{D}_1 \\ A_1 \end{array} \quad \begin{array}{c} n \in \mathbb{N} \\ \dots \end{array} \quad \begin{array}{c} \triangle \mathcal{D}_n \\ A_n \\ A_i^{x_i} \end{array} = \begin{array}{c} \triangle \mathcal{D}_i \\ A_i \end{array} \quad t(\mathcal{D}_1, \dots, \mathcal{D}_n) = \mathcal{D}_i$$

Rmk. In a derivation, different formulas are associated with different variables.
For several occurrences of A as hypotheses, we choose the same x or another variable.
 $\rightsquigarrow$ A variable represents a (possibly empty) **parcel of hypotheses** of the same formula.

A computational interpretation of ND: variable

Let the derivation $\mathcal{D} \triangleright_{\text{ND}} A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ consist of a single hypothesis $A_i^{x_i}$, that is,

$$\mathcal{D} = A_i^{x_i} \quad (\text{with } B = A_i)$$

The derivation $\mathcal{D}$ can then be seen a function

$$t = (x_1, \dots, x_n) \mapsto x_i \quad (\approx \text{projection})$$

represented by the **variable** x_i . Indeed, given the arguments $\mathcal{D}_1 \triangleright_{\text{ND}} \vdash A_1, \dots, \mathcal{D}_n \triangleright_{\text{ND}} \vdash A_n$,

$$\begin{array}{c} \triangle D_1 \\ A_1 \end{array} \quad \begin{array}{c} n \in \mathbb{N} \\ \vdots \\ \end{array} \quad \begin{array}{c} \triangle D_n \\ A_n \end{array} = \begin{array}{c} \triangle D_i \\ A_i \end{array} \quad t(\mathcal{D}_1, \dots, \mathcal{D}_n) = \mathcal{D}_i$$

Rmk. In a derivation, different formulas are associated with different variables.
For several occurrences of A as hypotheses, we choose the same x or another variable.
 $\rightsquigarrow$ A variable represents a (possibly empty) **parcel of hypotheses** of the same formula.

A computational interpretation of ND: variable

Let the derivation $\mathcal{D} \triangleright_{\text{ND}} A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ consist of a single hypothesis $A_i^{x_i}$, that is,

$$\mathcal{D} = A_i^{x_i} \quad (\text{with } B = A_i)$$

The derivation $\mathcal{D}$ can then be seen a function

$$t = (x_1, \dots, x_n) \mapsto x_i \quad (\approx \text{projection})$$

represented by the **variable** x_i . Indeed, given the arguments $\mathcal{D}_1 \triangleright_{\text{ND}} \vdash A_1, \dots, \mathcal{D}_n \triangleright_{\text{ND}} \vdash A_n$,

$$\begin{array}{c} \triangle D_1 \\ A_1 \end{array} \quad \begin{array}{c} n \in \mathbb{N} \\ \vdots \\ \end{array} \quad \begin{array}{c} \triangle D_n \\ A_n \end{array} = \begin{array}{c} \triangle D_i \\ A_i \end{array} \quad t(\mathcal{D}_1, \dots, \mathcal{D}_n) = \mathcal{D}_i$$

Rmk. In a derivation, different formulas are associated with different variables.
For several occurrences of A as hypotheses, we choose the same x or another variable.
 $\rightsquigarrow$ A variable represents a (possibly empty) **parcel of hypotheses** of the same formula.

A computational interpretation of ND: abstraction

Let the last rule of the derivation $\mathcal{D} \triangleright_{\text{ND}} A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ be $\Rightarrow_i^y$, that is,

$$\mathcal{D} = \frac{[C^y] \quad \begin{array}{c} A_1^{x_1} \quad \dots \quad A_n^{x_n} \\ \text{---} \\ \mathcal{D}' \\ \text{---} \\ A \end{array}}{C \Rightarrow A} \Rightarrow_i^y$$

(with $B = C \Rightarrow A$). Let

$$s = (y, x_1, \dots, x_n) \mapsto s(y, x_1, \dots, x_n)$$

be the function associated with the derivation $\mathcal{D}'$.

The derivation $\mathcal{D}$ can be seen as the function

$$t = (x_1, \dots, x_n) \mapsto \underbrace{(y \mapsto s(y, x_1, \dots, x_n))}_{(\lambda y.s)(x_1, \dots, x_n)} \quad (\approx \text{currying})$$

represented by the **abstraction** $\lambda y.s$. Indeed, given $\mathcal{D}_1 \triangleright_{\text{ND}} \vdash A_1, \dots, \mathcal{D}_n \triangleright_{\text{ND}} \vdash A_n$,

$$\begin{array}{c} \begin{array}{c} \mathcal{D}_1 \quad \dots \quad \mathcal{D}_n \\ \text{---} \\ A_1 \quad \dots \quad A_n \end{array} \\ [C^y] \quad \begin{array}{c} A_1^{x_1} \quad \dots \quad A_n^{x_n} \\ \text{---} \\ \mathcal{D}' \\ \text{---} \\ A \end{array} \\ \hline C \Rightarrow A \Rightarrow_i^y \end{array} = \begin{array}{c} \begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_n \\ \text{---} \\ A_1 \quad A_n \end{array} \\ [C^y] \quad \begin{array}{c} \mathcal{D}' \\ \text{---} \\ A \end{array} \\ \hline C \Rightarrow A \Rightarrow_i^y \end{array}$$

$$\begin{aligned} t(\mathcal{D}_1, \dots, \mathcal{D}_n) &= (y \mapsto s(y, \mathcal{D}_1, \dots, \mathcal{D}_n)) \\ &= \lambda y.s(y, \mathcal{D}_1, \dots, \mathcal{D}_n) \end{aligned}$$

(higher order)

A computational interpretation of ND: abstraction

Let the last rule of the derivation $\mathcal{D} \triangleright_{\text{ND}} A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ be $\Rightarrow_i^y$, that is,

$$\mathcal{D} = \frac{[C^y] \quad \begin{array}{c} A_1^{x_1} \quad \dots \quad A_n^{x_n} \\ \vdots \end{array} \quad \begin{array}{c} \diagup \\ \mathcal{D}' \\ \diagdown \end{array} \quad A}{C \Rightarrow A} \Rightarrow_i^y$$

(with $B = C \Rightarrow A$). Let

$$s = (y, x_1, \dots, x_n) \mapsto s(y, x_1, \dots, x_n)$$

be the function associated with the derivation $\mathcal{D}'$.

The derivation $\mathcal{D}$ can be seen as the function

$$t = (x_1, \dots, x_n) \mapsto \left(y \mapsto \overbrace{s(y, x_1, \dots, x_n)}^{(\lambda y.s)(x_1, \dots, x_n)} \right) \quad (\approx \text{currying})$$

represented by the abstraction $\lambda y.s$. Indeed, given $\mathcal{D}_1 \triangleright_{\text{ND}} \vdash A_1, \dots, \mathcal{D}_n \triangleright_{\text{ND}} \vdash A_n$,

$$\frac{[C^y] \quad \begin{array}{c} \begin{array}{c} \mathcal{D}_1 \\ \vdots \end{array} \quad \begin{array}{c} \mathcal{D}_n \\ \vdots \end{array} \\ A_1 \quad \dots \quad A_n \end{array} \quad \begin{array}{c} \diagup \\ \mathcal{D}' \\ \diagdown \end{array} \quad A}{C \Rightarrow A} \Rightarrow_i^y = \frac{[C^y] \quad \begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_n \\ \vdots \end{array} \quad \begin{array}{c} \diagup \\ \mathcal{D}' \\ \diagdown \end{array} \quad A}{C \Rightarrow A} \Rightarrow_i^y$$

$$\begin{aligned} t(\mathcal{D}_1, \dots, \mathcal{D}_n) &= (y \mapsto s(y, \mathcal{D}_1, \dots, \mathcal{D}_n)) \\ &= \lambda y.s(y, \mathcal{D}_1, \dots, \mathcal{D}_n) \end{aligned}$$

(higher order)

A computational interpretation of ND: application

Let the last rule of the derivation $\mathcal{D} \triangleright_{\text{ND}} A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ be $\Rightarrow_e$, that is,

$$\mathcal{D} = \frac{\begin{array}{c} A_1^{x_1} \quad \dots \quad A_n^{x_n} \\ \vdots \quad \vdots \quad \vdots \\ \mathcal{D}_1 \end{array} \quad \begin{array}{c} A_1^{x_1} \quad \dots \quad A_n^{x_n} \\ \vdots \quad \vdots \quad \vdots \\ \mathcal{D}_2 \end{array}}{A \Rightarrow B \quad A \Rightarrow_e B} \Rightarrow_e$$

Let s_1 and s_2 be the functions

$$s_1 = (x_1, \dots, x_n) \mapsto s_1(x_1, \dots, x_n)$$

$$s_2 = (x_1, \dots, x_n) \mapsto s_2(x_1, \dots, x_n)$$

associated with the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

The derivation $\mathcal{D}$ can be seen as the function t applying s_1 to s_2

$$t = (x_1, \dots, x_n) \mapsto s_1(x_1, \dots, x_n)(s_2(x_1, \dots, x_n)) \quad (\approx \text{application})$$

represented by the **juxtaposition** $s_1 s_2$. Indeed, given $\mathcal{D}_1 \triangleright_{\text{ND}} \vdash A_1, \dots, \mathcal{D}_n \triangleright_{\text{ND}} \vdash A_n$,

$$\begin{array}{c} \begin{array}{c} \mathcal{D}_1 \quad \dots \quad \mathcal{D}_n \\ \vdots \quad \vdots \quad \vdots \\ A_1 \quad \dots \quad A_n \end{array} \\ \vdots \quad \vdots \quad \vdots \\ A_1^{x_1} \quad \dots \quad A_n^{x_n} \\ \vdots \quad \vdots \quad \vdots \\ \mathcal{D}_1 \quad \mathcal{D}_2 \\ \vdots \quad \vdots \\ A \Rightarrow B \quad A \Rightarrow_e B \end{array} = \begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_n \quad \mathcal{D}_1 \quad \mathcal{D}_n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ A_1 \quad \dots \quad A_n \quad A_1 \quad \dots \quad A_n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ A \Rightarrow B \quad A \Rightarrow_e B \end{array}$$

$$t(\mathcal{D}_1, \dots, \mathcal{D}_n) = s_1(\mathcal{D}_1, \dots, \mathcal{D}_n)(s_2(\mathcal{D}_1, \dots, \mathcal{D}_n))$$

A computational interpretation of ND: application

Let the last rule of the derivation $\mathcal{D} \triangleright_{\text{ND}} A_1^{x_1}, \dots, A_n^{x_n} \vdash B$ be $\Rightarrow_e$, that is,

$$\mathcal{D} = \frac{\begin{array}{c} A_1^{x_1} \quad \dots \quad A_n^{x_n} \\ \vdots \quad \vdots \quad \vdots \\ \mathcal{D}_1 \end{array} \quad \begin{array}{c} A_1^{x_1} \quad \dots \quad A_n^{x_n} \\ \vdots \quad \vdots \quad \vdots \\ \mathcal{D}_2 \end{array}}{A \Rightarrow B \quad A} \Rightarrow_e B$$

Let s_1 and s_2 be the functions

$$s_1 = (x_1, \dots, x_n) \mapsto s_1(x_1, \dots, x_n)$$

$$s_2 = (x_1, \dots, x_n) \mapsto s_2(x_1, \dots, x_n)$$

associated with the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

The derivation $\mathcal{D}$ can be seen as the function t applying s_1 to s_2

$$t = (x_1, \dots, x_n) \mapsto s_1(x_1, \dots, x_n)(s_2(x_1, \dots, x_n)) \quad (\approx \text{application})$$

represented by the **juxtaposition** $s_1 s_2$. Indeed, given $\mathcal{D}_1 \triangleright_{\text{ND}} \vdash A_1, \dots, \mathcal{D}_n \triangleright_{\text{ND}} \vdash A_n$,

$$\begin{array}{c} \begin{array}{c} \mathcal{D}_1 \quad \dots \quad \mathcal{D}_n \\ \vdots \quad \vdots \quad \vdots \\ A_1 \quad \dots \quad A_n \end{array} \\ \begin{array}{c} A_1^{x_1} \quad \dots \quad A_n^{x_n} \\ \vdots \quad \vdots \quad \vdots \\ \mathcal{D}_1 \quad \mathcal{D}_2 \end{array} \\ \frac{A \Rightarrow B \quad A}{B} \Rightarrow_e \end{array} = \begin{array}{c} \begin{array}{c} \mathcal{D}_1 \quad \dots \quad \mathcal{D}_n \\ \vdots \quad \vdots \quad \vdots \\ A_1 \quad \dots \quad A_n \end{array} \\ \begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_2 \end{array} \\ \frac{A \Rightarrow B \quad A}{B} \Rightarrow_e \end{array}$$

$$t(\mathcal{D}_1, \dots, \mathcal{D}_n) = s_1(\mathcal{D}_1, \dots, \mathcal{D}_n)(s_2(\mathcal{D}_1, \dots, \mathcal{D}_n))$$

Summing up: decorating derivations in ND with terms

Given an infinite set of variables $x, y, z, \dots$, **terms** are built up on the grammar:

$$s, t ::= x \mid \lambda x. t \mid st$$

Each derivation $\mathcal{D}$ in ND is associated with a **unique** term $(\mathcal{D})_\lambda$ defined inductively by:

$$(A^x)_\lambda = x \quad \left(\frac{\begin{array}{c} [A^x] \\ \vdots \\ \mathcal{D} \\ B \end{array}}{A \Rightarrow B} \Rightarrow_i^x \right)_\lambda = \lambda x. (\mathcal{D})_\lambda \quad \left(\frac{\begin{array}{cc} \vdots \mathcal{D}_1 & \vdots \mathcal{D}_2 \\ A \Rightarrow B & A \Rightarrow_e \end{array}}{B} \Rightarrow_e \right)_\lambda = (\mathcal{D}_1)_\lambda (\mathcal{D}_2)_\lambda$$

Rmk. The variable labeling the hypotheses and $\Rightarrow_i$ is crucial to uniquely determine $(\mathcal{D})_\lambda$.

We have a **static correspondence** between derivations in ND and terms.

- We can **decorate** each formula occurrence in a derivation in ND with a term.
- For every derivation $\mathcal{D}$ in ND, its term $(\mathcal{D})_\lambda$ is the decoration of its conclusion.
- Terms are a concise (and linear) representations of derivations (trees) in ND.

Summing up: decorating derivations in ND with terms

Given an infinite set of variables $x, y, z, \dots$, **terms** are built up on the grammar:

$$s, t ::= x \mid \lambda x. t \mid st$$

Each derivation $\mathcal{D}$ in ND is associated with a **unique** term $(\mathcal{D})_\lambda$ defined inductively by:

$$(A^x)_\lambda = x \qquad \left(\frac{\begin{array}{c} [A^x] \\ \vdots \\ \mathcal{D} \\ B \end{array}}{A \Rightarrow B} \Rightarrow_i^x \right)_\lambda = \lambda x. (\mathcal{D})_\lambda \qquad \left(\frac{\begin{array}{cc} \vdots \mathcal{D}_1 & \vdots \mathcal{D}_2 \\ A \Rightarrow B & A \Rightarrow_e \end{array}}{B} \Rightarrow_e \right)_\lambda = (\mathcal{D}_1)_\lambda (\mathcal{D}_2)_\lambda$$

Rmk. The variable labeling the hypotheses and $\Rightarrow_i$ is crucial to uniquely determine $(\mathcal{D})_\lambda$.

We have a **static correspondence** between derivations in ND and terms.

- We can **decorate** each formula occurrence in a derivation in ND with a term.
- For every derivation $\mathcal{D}$ in ND, its term $(\mathcal{D})_\lambda$ is the decoration of its conclusion.
- Terms are a concise (and linear) representations of derivations (trees) in ND.

Summing up: decorating derivations in ND with terms

Given an infinite set of variables $x, y, z, \dots$, **terms** are built up on the grammar:

$$s, t ::= x \mid \lambda x. t \mid st$$

Each derivation $\mathcal{D}$ in ND is associated with a **unique** term $(\mathcal{D})_\lambda$ defined inductively by:

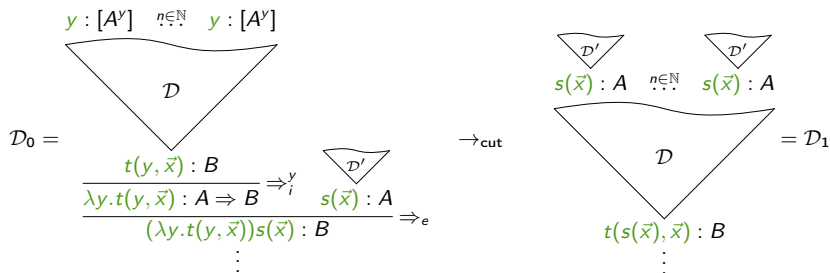
$$(A^x)_\lambda = x \quad \left(\frac{\begin{array}{c} [A^x] \\ \vdots \\ \mathcal{D} \\ B \end{array}}{A \Rightarrow B} \Rightarrow_i^x \right)_\lambda = \lambda x. (\mathcal{D})_\lambda \quad \left(\frac{\begin{array}{cc} \vdots \mathcal{D}_1 & \vdots \mathcal{D}_2 \\ A \Rightarrow B & A \Rightarrow_e \end{array}}{B} \Rightarrow_e \right)_\lambda = (\mathcal{D}_1)_\lambda (\mathcal{D}_2)_\lambda$$

Rmk. The variable labeling the hypotheses and $\Rightarrow_i$ is crucial to uniquely determine $(\mathcal{D})_\lambda$.

We have a **static correspondence** between derivations in ND and terms.

- We can **decorate** each formula occurrence in a derivation in ND with a term.
- For every derivation $\mathcal{D}$ in ND, its term $(\mathcal{D})_\lambda$ is the decoration of its conclusion.
- Terms are a concise (and linear) representations of derivations (trees) in ND.

Dynamics: cut-elimination as a computational step



Rmk. Cut-elimination explains why term juxtaposition can be seen as function application.

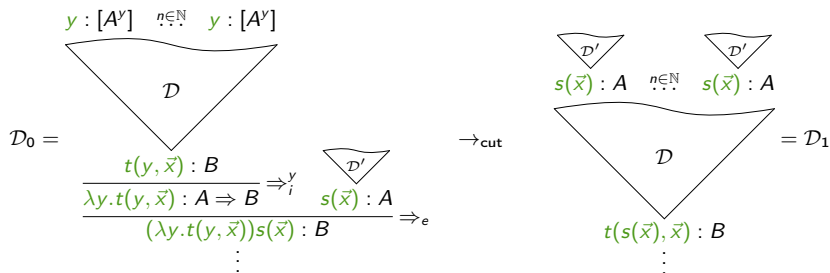
The decoration of derivations with terms **commutes** with cut-elimination via the step:

$$(\lambda y. t)s \rightarrow_{\beta} t\{s/y\}$$

where $t\{s/y\}$ stands for the **substitution** of s for the free occurrences of y in t .



Dynamics: cut-elimination as a computational step



Rmk. Cut-elimination explains why term juxtaposition can be seen as function application.

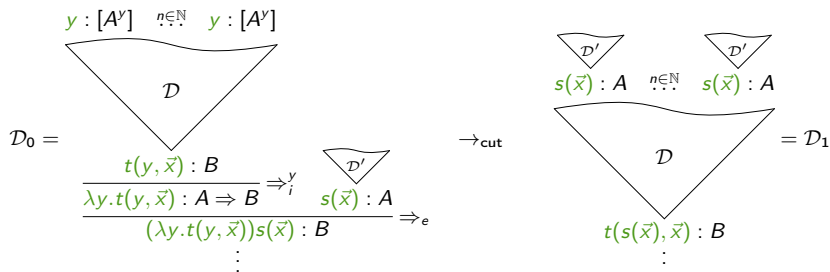
The decoration of derivations with terms **commutes** with cut-elimination via the step:

$$(\lambda y. t)s \rightarrow_{\beta} t\{s/y\}$$

where $t\{s/y\}$ stands for the **substitution** of s for the free occurrences of y in t .



Dynamics: cut-elimination as a computational step



Rmk. Cut-elimination explains why term juxtaposition can be seen as function application.

The decoration of derivations with terms **commutes** with cut-elimination via the step:

$$(\lambda y. t)s \rightarrow_{\beta} t\{s/y\}$$

where $t\{s/y\}$ stands for the **substitution** of s for the free occurrences of y in t .



Examples of decorations of derivations in ND

$$\begin{array}{c} x : A \\ \hline \lambda x. x : A \Rightarrow A \Rightarrow_i^x \end{array}$$

$$\begin{array}{c} \frac{x : [A^x]}{\lambda y. x : A \Rightarrow A \Rightarrow_i^y} \\ \hline \lambda x. \lambda y. x : A \Rightarrow A \Rightarrow A \Rightarrow_i^x \end{array}$$

$$\begin{array}{c} \frac{y : [A^y]}{\lambda y. y : A \Rightarrow A \Rightarrow_i^y} \\ \hline \lambda x. \lambda y. y : A \Rightarrow A \Rightarrow A \Rightarrow_i^x \end{array}$$

$$\begin{array}{c} \frac{x : [(A \Rightarrow B \Rightarrow C)^x] \quad z : [A^z]}{xz : B \Rightarrow C} \Rightarrow_e \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e \\ \hline \frac{(xz)(yz) : C}{\lambda z. (xz)(yz) : A \Rightarrow C} \Rightarrow_i^z \\ \hline \frac{\lambda y. \lambda z. (xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}{\lambda x. \lambda y. \lambda z. (xz)(yz) : (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^y \end{array}$$

Examples of decorations of derivations in ND

$$\frac{x : A \quad x : [A^x]}{\lambda x. x : A \Rightarrow A} \Rightarrow_i^x$$

$$\frac{\frac{x : [A^x]}{\lambda y. x : A \Rightarrow A} \Rightarrow_i^y}{\lambda x. \lambda y. x : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x$$

$$\frac{\frac{y : [A^y]}{\lambda y. y : A \Rightarrow A} \Rightarrow_i^y}{\lambda x. \lambda y. y : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x$$

$$\frac{\frac{x : [(A \Rightarrow B \Rightarrow C)^x] \quad z : [A^z]}{xz : B \Rightarrow C} \Rightarrow_e \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e}{\frac{(xz)(yz) : C}{\lambda z. (xz)(yz) : A \Rightarrow C} \Rightarrow_i^z} \Rightarrow_e$$

$$\frac{\lambda y. \lambda z. (xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}{\lambda x. \lambda y. \lambda z. (xz)(yz) : (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^y \Rightarrow_i^x$$

Examples of decorations of derivations in ND

$$\begin{array}{c}
 x : A \qquad \frac{x : [A^x]}{\lambda x.x : A \Rightarrow A} \Rightarrow_i^x \qquad \frac{\frac{x : [A^x]}{\lambda y.x : B \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.x : A \Rightarrow B \Rightarrow A} \Rightarrow_i^x
 \end{array}$$

$$\frac{\frac{x : [A^x]}{\lambda y.x : A \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.x : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \qquad \frac{\frac{y : [A^y]}{\lambda y.y : A \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.y : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x$$

$$\frac{x : [(A \Rightarrow B \Rightarrow C)^x] \quad z : [A^z]}{xz : B \Rightarrow C} \Rightarrow_e \qquad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e$$

$$\frac{\frac{(xz)(yz) : C}{\lambda z.(xz)(yz) : A \Rightarrow C} \Rightarrow_i^z}{\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^y$$

$$\frac{\lambda x.\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}{\lambda x.\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^x$$

Examples of decorations of derivations in ND

$$\begin{array}{c}
 x : A \quad \frac{x : [A^x]}{\lambda x.x : A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{x : [A^x]}{\lambda y.x : B \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.x : A \Rightarrow B \Rightarrow A} \Rightarrow_i^x
 \end{array}$$

$$\frac{\frac{x : [A^x]}{\lambda y.x : A \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.x : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{y : [A^y]}{\lambda y.y : A \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.y : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x$$

$$\frac{x : [(A \Rightarrow B \Rightarrow C)^x] \quad z : [A^z]}{xz : B \Rightarrow C} \Rightarrow_e \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e$$

$$\frac{\frac{(xz)(yz) : C}{\lambda z.(xz)(yz) : A \Rightarrow C} \Rightarrow_i^z}{\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^y$$

$$\frac{\lambda x.\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}{\lambda x.\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^x$$

Examples of decorations of derivations in ND

$$\begin{array}{c}
 x : A \qquad \frac{x : [A^x]}{\lambda x.x : A \Rightarrow A} \Rightarrow_i^x \qquad \frac{\frac{x : [A^x]}{\lambda y.x : B \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.x : A \Rightarrow B \Rightarrow A} \Rightarrow_i^x
 \end{array}$$

$$\frac{\frac{x : [A^x]}{\lambda y.x : A \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.x : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \qquad \frac{\frac{y : [A^y]}{\lambda y.y : A \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.y : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x$$

$$\frac{x : [(A \Rightarrow B \Rightarrow C)^x] \quad z : [A^z]}{xz : B \Rightarrow C} \Rightarrow_e \qquad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e$$

$$\frac{\frac{(xz)(yz) : C}{\lambda z.(xz)(yz) : A \Rightarrow C} \Rightarrow_i^z}{\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^y$$

$$\frac{\lambda x.\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}{\lambda x.\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^x$$

Examples of decorations of derivations in ND

$$\begin{array}{c}
 x : A \quad \frac{x : [A^x]}{\lambda x.x : A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{x : [A^x]}{\lambda y.x : B \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.x : A \Rightarrow B \Rightarrow A} \Rightarrow_i^x
 \\
 \\
 \frac{\frac{x : [A^x]}{\lambda y.x : A \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.x : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x \quad \frac{\frac{y : [A^y]}{\lambda y.y : A \Rightarrow A} \Rightarrow_i^y}{\lambda x.\lambda y.y : A \Rightarrow A \Rightarrow A} \Rightarrow_i^x
 \\
 \\
 \frac{x : [(A \Rightarrow B \Rightarrow C)^x] \quad z : [A^z]}{xz : B \Rightarrow C} \Rightarrow_e \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e
 \\
 \frac{\frac{\frac{(xz)(yz) : C}{\lambda z.(xz)(yz) : A \Rightarrow C} \Rightarrow_i^z}{\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^y}{\lambda x.\lambda y.\lambda z.(xz)(yz) : (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)} \Rightarrow_i^x
 \end{array}$$

Example of decorations of derivations in ND with cut-elimination

$$\frac{\frac{x : [(X \Rightarrow X)^x]}{\lambda x.x : (X \Rightarrow X) \Rightarrow X \Rightarrow X} \Rightarrow_i^x \quad \frac{y : [X^y]}{\lambda y.y : X \Rightarrow X} \Rightarrow_i^y}{(\lambda x.x)\lambda y.y : X \Rightarrow X} \Rightarrow_e \quad \xrightarrow{\text{cut}} \quad \frac{y : [X^y]}{\lambda y.y : X \Rightarrow X} \Rightarrow_i^y$$

Rmk. $(\lambda x.x)\lambda y.y \rightarrow_{\beta} x\{\lambda y.y/x\} = \lambda y.y \rightsquigarrow$ cut-elimination commutes with decoration.

Example of decorations of derivations in ND with cut-elimination

$$\frac{\frac{x : [(X \Rightarrow X)^x]}{\lambda x.x : (X \Rightarrow X) \Rightarrow X \Rightarrow X} \Rightarrow_i^x \quad \frac{y : [X^y]}{\lambda y.y : X \Rightarrow X} \Rightarrow_i^y}{(\lambda x.x)\lambda y.y : X \Rightarrow X} \Rightarrow_e \quad \xrightarrow{\text{cut}} \quad \frac{y : [X^y]}{\lambda y.y : X \Rightarrow X} \Rightarrow_i^y$$

Rmk. $(\lambda x.x)\lambda y.y \rightarrow_{\beta} x\{\lambda y.y/x\} = \lambda y.y \rightsquigarrow$ cut-elimination commutes with decoration.

Example of decorations of derivations in ND with cut-elimination

$$\frac{\frac{x : [(X \Rightarrow X)^x]}{\lambda x.x : (X \Rightarrow X) \Rightarrow X \Rightarrow X} \Rightarrow_i^x \quad \frac{y : [X^y]}{\lambda y.y : X \Rightarrow X} \Rightarrow_i^y}{(\lambda x.x)\lambda y.y : X \Rightarrow X} \Rightarrow_e \quad \xrightarrow{\text{cut}} \quad \frac{y : [X^y]}{\lambda y.y : X \Rightarrow X} \Rightarrow_i^y$$

Rmk. $(\lambda x.x)\lambda y.y \rightarrow_\beta x\{\lambda y.y/x\} = \lambda y.y \rightsquigarrow$ cut-elimination commutes with decoration.

Another example of decorations of derivations in ND with cut-elimination

$$\begin{array}{c}
 \frac{x : [(A \Rightarrow B \Rightarrow A)^x] \quad z : [A^z]}{xz : B \Rightarrow A} \Rightarrow_e \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e \\
 \frac{}{(xz)(yz) : A} \Rightarrow_e \\
 \frac{}{\lambda z. (xz)(yz) : A \Rightarrow A} \Rightarrow_i^z \\
 \frac{}{\lambda y. \lambda z. (xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow A)} \Rightarrow_i^y \\
 \frac{}{\lambda x. \lambda y. \lambda z. (xz)(yz) : (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow A} \Rightarrow_i^x \quad \frac{a : [A^a]}{\lambda b. a : B \Rightarrow A} \Rightarrow_i^b \\
 \frac{}{\lambda a. \lambda b. a : A \Rightarrow (B \Rightarrow A)} \Rightarrow_i^a \\
 \frac{}{(\lambda x. \lambda y. \lambda z. (xz)(yz)) \lambda a. \lambda b. a : (A \Rightarrow B) \Rightarrow A \Rightarrow A} \Rightarrow_e
 \end{array}$$

$$\begin{array}{c}
 \downarrow \text{cut} \\
 \frac{a : [A^a]}{\lambda b. a : B \Rightarrow A} \Rightarrow_i^b \\
 \frac{}{\lambda a. \lambda b. a : A \Rightarrow (B \Rightarrow A)} \Rightarrow_i^a \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e \\
 \frac{}{(\lambda a. \lambda b. a)z : B \Rightarrow A} \Rightarrow_e \\
 \frac{}{((\lambda a. \lambda b. a)z)(yz) : A} \Rightarrow_e \\
 \frac{}{\lambda z. ((\lambda a. \lambda b. a)z)(yz) : A \Rightarrow A} \Rightarrow_i^z \\
 \frac{}{\lambda y. \lambda z. ((\lambda a. \lambda b. a)z)(yz) : (A \Rightarrow B) \Rightarrow A \Rightarrow A} \Rightarrow_i^y
 \end{array}$$

Rmk. $(\lambda x. \lambda y. \lambda z. (xz)(yz)) \lambda a. \lambda b. a \rightarrow_\beta (\lambda y. \lambda z. (xz)(yz)) \{ \lambda a. \lambda b. a / x \} =$
 $\lambda y. \lambda z. ((\lambda a. \lambda b. a)z)(yz) \rightsquigarrow$ cut-elimination commutes with decoration.

Another example of decorations of derivations in ND with cut-elimination

$$\begin{array}{c}
 \frac{x : [(A \Rightarrow B \Rightarrow A)^x] \quad z : [A^z]}{xz : B \Rightarrow A} \Rightarrow_e \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e \\
 \hline
 \frac{(xz)(yz) : A}{\lambda z. (xz)(yz) : A \Rightarrow A} \Rightarrow_i^z \\
 \hline
 \frac{\lambda y. \lambda z. (xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow A)}{\lambda y. \lambda z. (xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow A)} \Rightarrow_i^y \\
 \hline
 \frac{\lambda x. \lambda y. \lambda z. (xz)(yz) : (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow A}{\lambda x. \lambda y. \lambda z. (xz)(yz) : (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow A} \Rightarrow_i^x \quad \frac{a : [A^a]}{\lambda b. a : B \Rightarrow A} \Rightarrow_i^b \\
 \hline
 \frac{\lambda x. \lambda y. \lambda z. (xz)(yz) : (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow A \quad \lambda a. \lambda b. a : A \Rightarrow (B \Rightarrow A)}{(\lambda x. \lambda y. \lambda z. (xz)(yz)) \lambda a. \lambda b. a : (A \Rightarrow B) \Rightarrow A \Rightarrow A} \Rightarrow_e
 \end{array}$$

$$\begin{array}{c}
 \downarrow \text{cut} \\
 \frac{a : [A^a]}{\lambda b. a : B \Rightarrow A} \Rightarrow_i^b \\
 \hline
 \frac{\lambda a. \lambda b. a : A \Rightarrow (B \Rightarrow A) \quad z : [A^z]}{(\lambda a. \lambda b. a)z : B \Rightarrow A} \Rightarrow_e \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e \\
 \hline
 \frac{((\lambda a. \lambda b. a)z)(yz) : A}{\lambda z. ((\lambda a. \lambda b. a)z)(yz) : A \Rightarrow A} \Rightarrow_i^z \\
 \hline
 \frac{\lambda y. \lambda z. ((\lambda a. \lambda b. a)z)(yz) : (A \Rightarrow B) \Rightarrow A \Rightarrow A}{\lambda y. \lambda z. ((\lambda a. \lambda b. a)z)(yz) : (A \Rightarrow B) \Rightarrow A \Rightarrow A} \Rightarrow_i^y
 \end{array}$$

Rmk. $(\lambda x. \lambda y. \lambda z. (xz)(yz)) \lambda a. \lambda b. a \rightarrow_\beta (\lambda y. \lambda z. (xz)(yz)) \{ \lambda a. \lambda b. a / x \} =$
 $\lambda y. \lambda z. ((\lambda a. \lambda b. a)z)(yz) \rightsquigarrow$ cut-elimination commutes with decoration.

Another example of decorations of derivations in ND with cut-elimination

$$\begin{array}{c}
 \frac{x : [(A \Rightarrow B \Rightarrow A)^x] \quad z : [A^z]}{xz : B \Rightarrow A} \Rightarrow_e \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e \\
 \hline
 \frac{(xz)(yz) : A}{\lambda z. (xz)(yz) : A \Rightarrow A} \Rightarrow_i^z \\
 \hline
 \frac{\lambda y. \lambda z. (xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow A)}{\lambda y. \lambda z. (xz)(yz) : (A \Rightarrow B) \Rightarrow (A \Rightarrow A)} \Rightarrow_i^y \\
 \hline
 \frac{\lambda x. \lambda y. \lambda z. (xz)(yz) : (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow A}{\lambda x. \lambda y. \lambda z. (xz)(yz) : (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow A} \Rightarrow_i^x \quad \frac{a : [A^a]}{\lambda b. a : B \Rightarrow A} \Rightarrow_i^b \\
 \hline
 \frac{\lambda x. \lambda y. \lambda z. (xz)(yz) : (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow A \quad \lambda a. \lambda b. a : A \Rightarrow (B \Rightarrow A)}{(\lambda x. \lambda y. \lambda z. (xz)(yz)) \lambda a. \lambda b. a : (A \Rightarrow B) \Rightarrow A \Rightarrow A} \Rightarrow_e
 \end{array}$$

$$\begin{array}{c}
 \downarrow \text{cut} \\
 \frac{a : [A^a]}{\lambda b. a : B \Rightarrow A} \Rightarrow_i^b \\
 \hline
 \frac{\lambda a. \lambda b. a : A \Rightarrow (B \Rightarrow A) \quad z : [A^z]}{(\lambda a. \lambda b. a)z : B \Rightarrow A} \Rightarrow_e \quad \frac{y : [(A \Rightarrow B)^y] \quad z : [A^z]}{yz : B} \Rightarrow_e \\
 \hline
 \frac{((\lambda a. \lambda b. a)z)(yz) : A}{\lambda z. ((\lambda a. \lambda b. a)z)(yz) : A \Rightarrow A} \Rightarrow_i^z \\
 \hline
 \frac{\lambda y. \lambda z. ((\lambda a. \lambda b. a)z)(yz) : (A \Rightarrow B) \Rightarrow A \Rightarrow A}{\lambda y. \lambda z. ((\lambda a. \lambda b. a)z)(yz) : (A \Rightarrow B) \Rightarrow A \Rightarrow A} \Rightarrow_i^y
 \end{array}$$

Rmk. $(\lambda x. \lambda y. \lambda z. (xz)(yz)) \lambda a. \lambda b. a \rightarrow_{\beta} (\lambda y. \lambda z. (xz)(yz)) \{ \lambda a. \lambda b. a / x \} =$
 $\lambda y. \lambda z. ((\lambda a. \lambda b. a)z)(yz) \rightsquigarrow$ cut-elimination commutes with decoration.

Curry-Howard and inverse decoration: from terms to derivations

So far, we have seen a first taste of the **Curry-Howard correspondence**:

- 1 derivations in ND (and ND_{seq}) can be decorated with λ -terms;
- 2 β -reduction (transforming λ -terms) mimics cut-elimination steps and vice versa.

Can we invert the process? Can derivations be associated with λ -terms?

Question. Given the term $\lambda f.\lambda x.fx$, what is the derivation associated with it?

$$\frac{\displaystyle \frac{\displaystyle \frac{f : [?^f] \quad x : [?^x]}{fx : ?} \Rightarrow_i^x \quad \lambda x.fx : ?}{\lambda f.\lambda x.fx : ?} \Rightarrow_i^f}{\lambda f.\lambda x.fx : ?} \Rightarrow_e$$

Problem. Without knowing the formulas associated with variables, there is no answer.

Question. Given the term $\lambda f^{X \Rightarrow X}.\lambda x^X.fx$, what is the derivation associated with it?

Question. Given the term $\lambda f^X.\lambda x^X.fx$, what is the derivation associated with it?

Rmk. Fixing formulas for variables (and hence for the whole term) is crucial!

Curry-Howard and inverse decoration: from terms to derivations

So far, we have seen a first taste of the **Curry-Howard correspondence**:

- 1 derivations in ND (and ND_{seq}) can be decorated with λ -terms;
- 2 β -reduction (transforming λ -terms) mimics cut-elimination steps and vice versa.

Can we invert the process? Can derivations be associated with λ -terms?

Question. Given the term $\lambda f.\lambda x.fx$, what is the derivation associated with it?

$$\frac{\displaystyle \frac{\displaystyle \frac{f : [?^f] \quad x : [?^x]}{fx : ?} \Rightarrow_i^x \quad \lambda x.fx : ?}{\lambda f.\lambda x.fx : ?} \Rightarrow_i^f}{\lambda f.\lambda x.fx : ?} \Rightarrow_e$$

Problem. Without knowing the formulas associated with variables, there is no answer.

Question. Given the term $\lambda f^{X \Rightarrow X}.\lambda x^X.fx$, what is the derivation associated with it?

Question. Given the term $\lambda f^X.\lambda x^X.fx$, what is the derivation associated with it?

Rmk. Fixing formulas for variables (and hence for the whole term) is crucial!

Curry-Howard and inverse decoration: from terms to derivations

So far, we have seen a first taste of the **Curry-Howard correspondence**:

- 1 derivations in ND (and ND_{seq}) can be decorated with λ -terms;
- 2 β -reduction (transforming λ -terms) mimics cut-elimination steps and vice versa.

Can we invert the process? Can derivations be associated with λ -terms?

Question. Given the term $\lambda f.\lambda x.fx$, what is the derivation associated with it?

$$\frac{\displaystyle \frac{\displaystyle \frac{f : [?^f] \quad x : [?^x]}{fx : ?} \Rightarrow_i^x \quad \lambda x.fx : ?}{\lambda f.\lambda x.fx : ?} \Rightarrow_i^f \Rightarrow_e$$

Problem. Without knowing the formulas associated with variables, there is no answer.

Question. Given the term $\lambda f^X.\lambda x^X.fx$, what is the derivation associated with it?

Question. Given the term $\lambda f^X.\lambda x^X.fx$, what is the derivation associated with it?

Rmk. Fixing formulas for variables (and hence for the whole term) is crucial!

Curry-Howard and inverse decoration: from terms to derivations

So far, we have seen a first taste of the **Curry-Howard correspondence**:

- 1 derivations in ND (and ND_{seq}) can be decorated with λ -terms;
- 2 β -reduction (transforming λ -terms) mimics cut-elimination steps and vice versa.

Can we invert the process? Can derivations be associated with λ -terms?

Question. Given the term $\lambda f.\lambda x.fx$, what is the derivation associated with it?

$$\frac{\displaystyle \frac{\displaystyle \frac{f : [?^f] \quad x : [?^x]}{fx : ?} \Rightarrow_i^x \quad \lambda x.fx : ?}{\lambda f.\lambda x.fx : ?} \Rightarrow_i^f \Rightarrow_e$$

Problem. Without knowing the formulas associated with variables, there is no answer.

Question. Given the term $\lambda f^{X \Rightarrow X}.\lambda x^X.fx$, what is the derivation associated with it?

Question. Given the term $\lambda f^X.\lambda x^X.fx$, what is the derivation associated with it?

Rmk. Fixing formulas for variables (and hence for the whole term) is crucial!

Curry-Howard and inverse decoration: from terms to derivations

So far, we have seen a first taste of the **Curry-Howard correspondence**:

- 1 derivations in ND (and ND_{seq}) can be decorated with λ -terms;
- 2 β -reduction (transforming λ -terms) mimics cut-elimination steps and vice versa.

Can we invert the process? Can derivations be associated with λ -terms?

Question. Given the term $\lambda f.\lambda x.fx$, what is the derivation associated with it?

$$\frac{\displaystyle \frac{\displaystyle \frac{f : [?^f] \quad x : [?^x]}{fx : ?} \Rightarrow_i^x \quad \lambda x.fx : ?}{\lambda f.\lambda x.fx : ?} \Rightarrow_i^f}{\lambda f.\lambda x.fx : ?} \Rightarrow_e$$

Problem. Without knowing the formulas associated with variables, there is no answer.

Question. Given the term $\lambda f^{X \Rightarrow X}.\lambda x^X.fx$, what is the derivation associated with it?

Question. Given the term $\lambda f^X.\lambda x^X.fx$, what is the derivation associated with it?

Rmk. Fixing formulas for variables (and hence for the whole term) is crucial!

Outline

- 1 Natural deduction for minimal logic, revisited
- 2 The Curry-Howard correspondence, informally
- 3 The simply typed λ -calculus
- 4 Conclusion, exercises and bibliography

The simply typed λ -calculus in Curry-style: typing rules

Types: $A, B ::= X \mid A \Rightarrow B$ (given a set of **ground** types ranged over by $X, Y, Z \dots$)
(λ -)Terms: $s, t ::= x \mid \lambda x. t \mid st$ (called variable, abstraction, application, respectively)

Def. An **environment** is a partial function with finite domain from variables to types. An environment Γ is denoted by $x_1 : A_1, \dots, x_n : A_n$ if $\text{dom}(\Gamma) = \{x_1, \dots, x_n\}$ and $\Gamma(x_i) = A_i$. A **judgment** is a triple $\Gamma \vdash t : A$ where Γ is an environment, t is a term, A is a type.

Def. A **derivation** in STLC is a finite tree built up from the typing rules below.

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Notation: $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ means that $\mathcal{D}$ is a derivation in STLC with conclusion $\Gamma \vdash t : A$. $\Gamma \vdash_{\text{STLC}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in STLC; we then say that $\Gamma \vdash t : A$ is **derivable** in STLC. A term t is **typable** in STLC if $\Gamma \vdash_{\text{STLC}} t : A$ for some Γ, A .

Ex. The terms x , $\lambda x. x$, xy and $\lambda x. \lambda y. xy$ are typable in STLC; xx and $\lambda x. xx$ are not.

Rmk. Types are the formulas of minimal logic. Derivations in STLC are the ones in ND_{seq} plus a decorating λ -term. A redex in ND_{seq} corresponds to a β -redex in their λ -term.

The simply typed λ -calculus in Curry-style: typing rules

Types: $A, B ::= X \mid A \Rightarrow B$ (given a set of **ground** types ranged over by $X, Y, Z \dots$)
(λ -)Terms: $s, t ::= x \mid \lambda x. t \mid st$ (called variable, abstraction, application, respectively)

Def. An **environment** is a partial function with finite domain from variables to types. An environment Γ is denoted by $x_1 : A_1, \dots, x_n : A_n$ if $\text{dom}(\Gamma) = \{x_1, \dots, x_n\}$ and $\Gamma(x_i) = A_i$. A **judgment** is a triple $\Gamma \vdash t : A$ where Γ is an environment, t is a term, A is a type.

Def. A **derivation** in STLC is a finite tree built up from the typing rules below.

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Notation: $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ means that $\mathcal{D}$ is a derivation in STLC with conclusion $\Gamma \vdash t : A$. $\Gamma \vdash_{\text{STLC}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in STLC; we then say that $\Gamma \vdash t : A$ is **derivable** in STLC. A term t is **typable** in STLC if $\Gamma \vdash_{\text{STLC}} t : A$ for some Γ, A .

Ex. The terms x , $\lambda x. x$, xy and $\lambda x. \lambda y. xy$ are typable in STLC; xx and $\lambda x. xx$ are not.

Rmk. Types are the formulas of minimal logic. Derivations in STLC are the ones in ND_{seq} plus a decorating λ -term. A redex in ND_{seq} corresponds to a β -redex in their λ -term.

The simply typed λ -calculus in Curry-style: typing rules

Types: $A, B ::= X \mid A \Rightarrow B$ (given a set of **ground** types ranged over by $X, Y, Z \dots$)
(λ -)Terms: $s, t ::= x \mid \lambda x.t \mid st$ (called variable, abstraction, application, respectively)

Def. An **environment** is a partial function with finite domain from variables to types. An environment Γ is denoted by $x_1 : A_1, \dots, x_n : A_n$ if $\text{dom}(\Gamma) = \{x_1, \dots, x_n\}$ and $\Gamma(x_i) = A_i$. A **judgment** is a triple $\Gamma \vdash t : A$ where Γ is an environment, t is a term, A is a type.

Def. A **derivation** in STLC is a finite tree built up from the typing rules below.

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Notation: $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ means that $\mathcal{D}$ is a derivation in STLC with conclusion $\Gamma \vdash t : A$. $\Gamma \vdash_{\text{STLC}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in STLC; we then say that $\Gamma \vdash t : A$ is **derivable** in STLC. A term t is **typable** in STLC if $\Gamma \vdash_{\text{STLC}} t : A$ for some Γ, A .

Ex. The terms x , $\lambda x.x$, xy and $\lambda x.\lambda y.xy$ are typable in STLC; xx and $\lambda x.xx$ are not.

Rmk. Types are the formulas of minimal logic. Derivations in STLC are the ones in ND_{seq} plus a decorating λ -term. A redex in ND_{seq} corresponds to a β -redex in their λ -term.

The simply typed λ -calculus in Curry-style: typing rules

Types: $A, B ::= X \mid A \Rightarrow B$ (given a set of **ground** types ranged over by $X, Y, Z \dots$)
(λ -)Terms: $s, t ::= x \mid \lambda x. t \mid st$ (called variable, abstraction, application, respectively)

Def. An **environment** is a partial function with finite domain from variables to types. An environment Γ is denoted by $x_1 : A_1, \dots, x_n : A_n$ if $\text{dom}(\Gamma) = \{x_1, \dots, x_n\}$ and $\Gamma(x_i) = A_i$. A **judgment** is a triple $\Gamma \vdash t : A$ where Γ is an environment, t is a term, A is a type.

Def. A **derivation** in STLC is a finite tree built up from the typing rules below.

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Notation: $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ means that $\mathcal{D}$ is a derivation in STLC with conclusion $\Gamma \vdash t : A$. $\Gamma \vdash_{\text{STLC}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in STLC; we then say that $\Gamma \vdash t : A$ is **derivable** in STLC. A term t is **typable** in STLC if $\Gamma \vdash_{\text{STLC}} t : A$ for some Γ, A .

Ex. The terms x , $\lambda x. x$, xy and $\lambda x. \lambda y. xy$ are typable in STLC; xx and $\lambda x. xx$ are not.

Rmk. Types are the formulas of minimal logic. Derivations in STLC are the ones in ND_{seq} plus a decorating λ -term. A redex in ND_{seq} corresponds to a β -redex in their λ -term.

The simply typed λ -calculus in Curry-style: free variables and β -reduction

In $\lambda x.t$, the λx is a **binder** for the occurrences of x in t not in the scope of another λx .

Def. The **free variables** of a term t are the variables that are not bound to a λ . Formally,

$$\text{fv}(x) = \{x\} \quad \text{fv}(st) = \text{fv}(s) \cup \text{fv}(t) \quad \text{fv}(\lambda x.t) = \text{fv}(t) \setminus \{x\}.$$

Rmk. Let $t = (z(\lambda x.\lambda y.xy))x$. Then $\text{fv}(t) = \{x, z\}$. Note that the **red** occurrence of x is bound to λx , but the **blue** occurrence of x is unbound, thus $x \in \text{fv}(t)$.

Proposition

- 1 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\Gamma, x : B \vdash_{\text{STLC}} t : A$, for any type B and variable $x \notin \text{dom}(\Gamma)$.
- 2 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\text{fv}(t) \subseteq \text{dom}(\Gamma)$ and $\Gamma|_{\text{fv}(t)} \vdash_{\text{STLC}} t : A$.

Moral. If $\Gamma \vdash_{\text{STLC}} t : A$, then Γ is essentially a type assignment for $\text{fv}(t)$.

β -reduction $\rightarrow_\beta$ is a binary relation on λ -terms generated by:

$$(\lambda x.t)s \rightarrow_\beta t\{s/x\}$$

where $t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t .

The simply typed λ -calculus in Curry-style: free variables and β -reduction

In $\lambda x.t$, the λx is a **binder** for the occurrences of x in t not in the scope of another λx .

Def. The **free variables** of a term t are the variables that are not bound to a λ . Formally,

$$\text{fv}(x) = \{x\} \quad \text{fv}(st) = \text{fv}(s) \cup \text{fv}(t) \quad \text{fv}(\lambda x.t) = \text{fv}(t) \setminus \{x\}.$$

Rmk. Let $t = (z(\lambda x.\lambda y.xy))x$. Then $\text{fv}(t) = \{x, z\}$. Note that the **red** occurrence of x is bound to λx , but the **blue** occurrence of x is unbound, thus $x \in \text{fv}(t)$.

Proposition

- 1 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\Gamma, x : B \vdash_{\text{STLC}} t : A$, for any type B and variable $x \notin \text{dom}(\Gamma)$.
- 2 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\text{fv}(t) \subseteq \text{dom}(\Gamma)$ and $\Gamma|_{\text{fv}(t)} \vdash_{\text{STLC}} t : A$.

Moral. If $\Gamma \vdash_{\text{STLC}} t : A$, then Γ is essentially a type assignment for $\text{fv}(t)$.

β -reduction $\rightarrow_\beta$ is a binary relation on λ -terms generated by:

$$(\lambda x.t)s \rightarrow_\beta t\{s/x\}$$

where $t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t .

The simply typed λ -calculus in Curry-style: free variables and β -reduction

In $\lambda x.t$, the λx is a **binder** for the occurrences of x in t not in the scope of another λx .

Def. The **free variables** of a term t are the variables that are not bound to a λ . Formally,

$$\text{fv}(x) = \{x\} \quad \text{fv}(st) = \text{fv}(s) \cup \text{fv}(t) \quad \text{fv}(\lambda x.t) = \text{fv}(t) \setminus \{x\}.$$

Rmk. Let $t = (z(\lambda x.\lambda y.xy))x$. Then $\text{fv}(t) = \{x, z\}$. Note that the **red** occurrence of x is bound to λx , but the **blue** occurrence of x is unbound, thus $x \in \text{fv}(t)$.

Proposition

- 1 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\Gamma, x : B \vdash_{\text{STLC}} t : A$, for any type B and variable $x \notin \text{dom}(\Gamma)$.
- 2 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\text{fv}(t) \subseteq \text{dom}(\Gamma)$ and $\Gamma|_{\text{fv}(t)} \vdash_{\text{STLC}} t : A$.

Moral. If $\Gamma \vdash_{\text{STLC}} t : A$, then Γ is essentially a type assignment for $\text{fv}(t)$.

β -reduction $\rightarrow_\beta$ is a binary relation on λ -terms generated by:

$$(\lambda x.t)s \rightarrow_\beta t\{s/x\}$$

where $t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t .

The simply typed λ -calculus in Curry-style: free variables and β -reduction

In $\lambda x.t$, the λx is a **binder** for the occurrences of x in t not in the scope of another λx .

Def. The **free variables** of a term t are the variables that are not bound to a λ . Formally,

$$\text{fv}(x) = \{x\} \quad \text{fv}(st) = \text{fv}(s) \cup \text{fv}(t) \quad \text{fv}(\lambda x.t) = \text{fv}(t) \setminus \{x\}.$$

Rmk. Let $t = (z(\lambda x.\lambda y.xy))x$. Then $\text{fv}(t) = \{x, z\}$. Note that the **red** occurrence of x is bound to λx , but the **blue** occurrence of x is unbound, thus $x \in \text{fv}(t)$.

Proposition

- 1 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\Gamma, x : B \vdash_{\text{STLC}} t : A$, for any type B and variable $x \notin \text{dom}(\Gamma)$.
- 2 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\text{fv}(t) \subseteq \text{dom}(\Gamma)$ and $\Gamma \upharpoonright_{\text{fv}(t)} \vdash_{\text{STLC}} t : A$.

Moral. If $\Gamma \vdash_{\text{STLC}} t : A$, then Γ is essentially a type assignment for $\text{fv}(t)$.

β -reduction $\rightarrow_\beta$ is a binary relation on λ -terms generated by:

$$(\lambda x.t)s \rightarrow_\beta t\{s/x\}$$

where $t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t .

The simply typed λ -calculus in Curry-style: free variables and β -reduction

In $\lambda x.t$, the λx is a **binder** for the occurrences of x in t not in the scope of another λx .

Def. The **free variables** of a term t are the variables that are not bound to a λ . Formally,

$$\text{fv}(x) = \{x\} \quad \text{fv}(st) = \text{fv}(s) \cup \text{fv}(t) \quad \text{fv}(\lambda x.t) = \text{fv}(t) \setminus \{x\}.$$

Rmk. Let $t = (z(\lambda x.\lambda y.xy))x$. Then $\text{fv}(t) = \{x, z\}$. Note that the **red** occurrence of x is bound to λx , but the **blue** occurrence of x is unbound, thus $x \in \text{fv}(t)$.

Proposition

- 1 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\Gamma, x : B \vdash_{\text{STLC}} t : A$, for any type B and variable $x \notin \text{dom}(\Gamma)$.
- 2 If $\Gamma \vdash_{\text{STLC}} t : A$, then $\text{fv}(t) \subseteq \text{dom}(\Gamma)$ and $\Gamma \upharpoonright_{\text{fv}(t)} \vdash_{\text{STLC}} t : A$.

Moral. If $\Gamma \vdash_{\text{STLC}} t : A$, then Γ is essentially a type assignment for $\text{fv}(t)$.

β -reduction $\rightarrow_\beta$ is a binary relation on λ -terms generated by:

$$(\lambda x.t)s \rightarrow_\beta t\{s/x\}$$

where $t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t .

The capture-avoiding substitution

Naive substitution $t[s/x]$: replacement of the free occurrences of the variable x in t by s .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t[s/x] = \lambda y.y(yy)$.

Problem: The free variable y in s has been **captured** by the λ in t . $\rightsquigarrow$ Undesirable.

Solution: **Capture-avoiding** substitution $t\{s/x\}$

- 1 first rename the bound variables in t with variables that do not occur in t or s ;
- 2 then perform the substitution of s for the free occurrences of x in t , defined as:

$$x\{s/x\} = s \quad y\{s/x\} = y \quad (\lambda y.t)\{s/x\} = \lambda y.(t\{s/x\}) \quad (t' t)\{s/x\} = t'\{s/x\} t\{s/x\}$$

$\rightsquigarrow$ So, the free variables of s are **not captured** by the λ 's in t .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t\{s/x\} = \lambda z.z(yy) = \lambda x.x(yy) \neq \lambda y.y(yy)$.

Rmk: The operation of renaming the bound variables in a term is called **α -equivalence**.

$\rightsquigarrow$ Capture-avoiding substitution makes sense, as we identify terms up to α -equivalence.

Idea. A bound variable occurrence is muted, it lose its identity and can be renamed.

The capture-avoiding substitution

Naive substitution $t[s/x]$: replacement of the free occurrences of the variable x in t by s .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t[s/x] = \lambda y.y(yy)$.

Problem: The free variable y in s has been **captured** by the λ in t . $\rightsquigarrow$ Undesirable.

Solution: **Capture-avoiding** substitution $t\{s/x\}$

- 1 first rename the bound variables in t with variables that do not occur in t or s ;
- 2 then perform the substitution of s for the free occurrences of x in t , defined as:

$$x\{s/x\} = s \quad y\{s/x\} = y \quad (\lambda y.t)\{s/x\} = \lambda y.(t\{s/x\}) \quad (t' t)\{s/x\} = t'\{s/x\} t\{s/x\}$$

$\rightsquigarrow$ So, the free variables of s are **not captured** by the λ 's in t .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t\{s/x\} = \lambda z.z(yy) = \lambda x.x(yy) \neq \lambda y.y(yy)$.

Rmk: The operation of renaming the bound variables in a term is called **α -equivalence**.

$\rightsquigarrow$ Capture-avoiding substitution makes sense, as we identify terms up to α -equivalence.

Idea. A bound variable occurrence is muted, it lose its identity and can be renamed.

The capture-avoiding substitution

Naive substitution $t[s/x]$: replacement of the free occurrences of the variable x in t by s .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t[s/x] = \lambda y.y(yy)$.

Problem: The free variable y in s has been **captured** by the λ in t . $\rightsquigarrow$ Undesirable.

Solution: **Capture-avoiding** substitution $t\{s/x\}$

- 1 first rename the bound variables in t with variables that do not occur in t or s ;
- 2 then perform the substitution of s for the free occurrences of x in t , defined as:

$$x\{s/x\} = s \quad y\{s/x\} = y \quad (\lambda y.t)\{s/x\} = \lambda y.(t\{s/x\}) \quad (t' t)\{s/x\} = t'\{s/x\}t\{s/x\}$$

$\rightsquigarrow$ So, the free variables of s are **not captured** by the λ 's in t .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t\{s/x\} = \lambda z.z(yy) = \lambda x.x(yy) \neq \lambda y.y(yy)$.

Rmk: The operation of renaming the bound variables in a term is called **α -equivalence**.

$\rightsquigarrow$ Capture-avoiding substitution makes sense, as we identify terms up to α -equivalence.

Idea. A bound variable occurrence is muted, it lose its identity and can be renamed.

The capture-avoiding substitution

Naive substitution $t[s/x]$: replacement of the free occurrences of the variable x in t by s .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t[s/x] = \lambda y.y(yy)$.

Problem: The free variable y in s has been **captured** by the λ in t . $\rightsquigarrow$ Undesirable.

Solution: **Capture-avoiding** substitution $t\{s/x\}$

- 1 first rename the bound variables in t with variables that do not occur in t or s ;
- 2 then perform the substitution of s for the free occurrences of x in t , defined as:

$$x\{s/x\} = s \quad y\{s/x\} = y \quad (\lambda y.t)\{s/x\} = \lambda y.(t\{s/x\}) \quad (t' t)\{s/x\} = t'\{s/x\} t\{s/x\}$$

$\rightsquigarrow$ So, the free variables of s are **not captured** by the λ 's in t .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t\{s/x\} = \lambda z.z(yy) = \lambda x.x(yy) \neq \lambda y.y(yy)$.

Rmk: The operation of renaming the bound variables in a term is called **α -equivalence**.

$\rightsquigarrow$ Capture-avoiding substitution makes sense, as we identify terms up to α -equivalence.

Idea. A bound variable occurrence is muted, it lose its identity and can be renamed.

The capture-avoiding substitution

Naive substitution $t[s/x]$: replacement of the free occurrences of the variable x in t by s .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t[s/x] = \lambda y.y(yy)$.

Problem: The free variable y in s has been **captured** by the λ in t . $\rightsquigarrow$ Undesirable.

Solution: **Capture-avoiding** substitution $t\{s/x\}$

- ➊ first rename the bound variables in t with variables that do not occur in t or s ;
- ➋ then perform the substitution of s for the free occurrences of x in t , defined as:

$$x\{s/x\} = s \quad y\{s/x\} = y \quad (\lambda y.t)\{s/x\} = \lambda y.(t\{s/x\}) \quad (t' t)\{s/x\} = t'\{s/x\}t\{s/x\}$$

$\rightsquigarrow$ So, the free variables of s are **not captured** by the λ 's in t .

Ex: Let $t = \lambda y.yx$ and $s = yy$. Then, $t\{s/x\} = \lambda z.z(yy) = \lambda x.x(yy) \neq \lambda y.y(yy)$.

Rmk: The operation of renaming the bound variables in a term is called **α -equivalence**.

$\rightsquigarrow$ Capture-avoiding substitution makes sense, as we identify terms up to α -equivalence.

Idea. A bound variable occurrence is muted, it lose its identity and can be renamed.

Some remarks about the simply typed λ -calculus in Curry-style

Rmk. The search for a derivation is uniquely determined by the term (**syntax-directed**).

↪ To build a derivation $\mathcal{D}$ of $\Gamma \vdash t : A$, just look at t to know the last rule of $\mathcal{D}$ (if any).

The types used in the simply typed λ -calculus are exactly the formulas of **minimal logic**.

The inference rules for the simply typed λ -calculus are the ones of ND_{seq} plus decoration.

↪ Every derivation in $\text{ND}/\text{ND}_{\text{seq}}$ corresponds to a **unique λ -term** typed in Curry-style.

Question: With every typable term in Curry-style is it associated a unique derivation? No!

$$\frac{\overline{x : X \vdash x : X}^{\text{var}}}{\vdash \lambda x.x : X \Rightarrow X}^{\lambda}$$

$$\frac{\overline{x : X \Rightarrow X \vdash x : X \Rightarrow X}^{\text{var}}}{\vdash \lambda x.x : (X \Rightarrow X) \Rightarrow X \Rightarrow X}^{\lambda}$$

↪ The map from $\text{ND}/\text{ND}_{\text{seq}}$ derivations to typable terms in Curry-style is **not injective**!

Idea: In Curry-style, types are extrinsic to terms (**dynamic** typing, *a posteriori*)

↪ Let us make them intrinsic to terms (**static** typing, *a priori*): **Church-style**.

Some remarks about the simply typed λ -calculus in Curry-style

Rmk. The search for a derivation is uniquely determined by the term (**syntax-directed**).

↪ To build a derivation $\mathcal{D}$ of $\Gamma \vdash t : A$, just look at t to know the last rule of $\mathcal{D}$ (if any).

The types used in the simply typed λ -calculus are exactly the formulas of **minimal logic**.
The inference rules for the simply typed λ -calculus are the ones of **ND_{seq}** plus decoration.

↪ Every derivation in ND/ND_{seq} corresponds to a **unique λ -term** typed in Curry-style.

Question: With every typable term in Curry-style is it associated a unique derivation? No!

$$\frac{\overline{x : X \vdash x : X}^{\text{var}}}{\vdash \lambda x.x : X \Rightarrow X}^{\lambda} \qquad \frac{\overline{x : X \Rightarrow X \vdash x : X \Rightarrow X}^{\text{var}}}{\vdash \lambda x.x : (X \Rightarrow X) \Rightarrow X \Rightarrow X}^{\lambda}$$

↪ The map from ND/ND_{seq} derivations to typable terms in Curry-style is **not injective**!

Idea: In Curry-style, types are extrinsic to terms (**dynamic** typing, *a posteriori*)

↪ Let us make them intrinsic to terms (**static** typing, *a priori*): **Church-style**.

Some remarks about the simply typed λ -calculus in Curry-style

Rmk. The search for a derivation is uniquely determined by the term (**syntax-directed**).

↪ To build a derivation $\mathcal{D}$ of $\Gamma \vdash t : A$, just look at t to know the last rule of $\mathcal{D}$ (if any).

The types used in the simply typed λ -calculus are exactly the formulas of **minimal logic**.
The inference rules for the simply typed λ -calculus are the ones of **ND_{seq}** plus decoration.

↪ Every derivation in ND/ND_{seq} corresponds to a **unique λ -term** typed in Curry-style.

Question: With every typable term in Curry-style is it associated a unique derivation? No!

$$\frac{\overline{x : X \vdash x : X}^{\text{var}}}{\vdash \lambda x.x : X \Rightarrow X}^{\lambda} \qquad \frac{\overline{x : X \Rightarrow X \vdash x : X \Rightarrow X}^{\text{var}}}{\vdash \lambda x.x : (X \Rightarrow X) \Rightarrow X \Rightarrow X}^{\lambda}$$

↪ The map from ND/ND_{seq} derivations to typable terms in Curry-style is **not injective**!

Idea: In Curry-style, types are extrinsic to terms (**dynamic** typing, *a posteriori*)

↪ Let us make them intrinsic to terms (**static** typing, *a priori*): **Church-style**.

Some remarks about the simply typed λ -calculus in Curry-style

Rmk. The search for a derivation is uniquely determined by the term (**syntax-directed**).

↪ To build a derivation $\mathcal{D}$ of $\Gamma \vdash t : A$, just look at t to know the last rule of $\mathcal{D}$ (if any).

The types used in the simply typed λ -calculus are exactly the formulas of **minimal logic**.
The inference rules for the simply typed λ -calculus are the ones of **ND_{seq}** plus decoration.

↪ Every derivation in ND/ND_{seq} corresponds to a **unique λ -term** typed in Curry-style.

Question: With every typable term in Curry-style is it associated a unique derivation? No!

$$\frac{\overline{x : X \vdash x : X}^{\text{var}}}{\vdash \lambda x.x : X \Rightarrow X}^{\lambda} \qquad \frac{\overline{x : X \Rightarrow X \vdash x : X \Rightarrow X}^{\text{var}}}{\vdash \lambda x.x : (X \Rightarrow X) \Rightarrow X \Rightarrow X}^{\lambda}$$

↪ The map from ND/ND_{seq} derivations to typable terms in Curry-style is **not injective**!

Idea: In Curry-style, types are extrinsic to terms (**dynamic** typing, *a posteriori*)

↪ Let us make them intrinsic to terms (**static** typing, *a priori*): **Church-style**.

The simply typed λ -calculus in Church-style

Idea. Let us make types intrinsic to terms (static typing, *a priori*): Church-style.

↪ Every abstracted variable in a term is associated with some type.

(λ -)Terms: $s, t ::= x \mid \lambda x^A.t \mid st$ (where A is any type, as defined for Curry-style)

Environments and judgments are as in Curry-style. The typing rules of Church-style are:

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Derivability of a judgment and typability of a λ -term are defined as in Curry-style.

β -reduction ($t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t):

$$(\lambda x^A.t)s \rightarrow_{\beta} t\{s/x\}$$

Rmk. Syntax-directedness and proposition on p. 21 hold true in Church-style as well.

Notations. $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$ means that $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in Curry/Church-style.

$\Gamma \vdash_{\text{Curry/Church}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$.

$\vdash_{\text{STLC}}$ (resp. $\triangleright_{\text{STLC}}$) means $\vdash_{\text{Curry}}$ or $\vdash_{\text{Church}}$ (resp. $\triangleright_{\text{Curry}}$ or $\triangleright_{\text{Church}}$) indifferently.

The simply typed λ -calculus in Church-style

Idea. Let us make types intrinsic to terms (static typing, *a priori*): Church-style.

$\rightsquigarrow$ Every abstracted variable in a term is associated with some type.

(λ -)Terms: $s, t ::= x \mid \lambda x^A.t \mid st$ (where A is any type, as defined for Curry-style)

Environments and judgments are as in Curry-style. The typing rules of Church-style are:

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Derivability of a judgment and typability of a λ -term are defined as in Curry-style.

β -reduction ($t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t):

$$(\lambda x^A.t)s \rightarrow_{\beta} t\{s/x\}$$

Rmk. Syntax-directedness and proposition on p. 21 hold true in Church-style as well.

Notations. $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$ means that $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in Curry/Church-style.

$\Gamma \vdash_{\text{Curry/Church}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$.

$\vdash_{\text{STLC}}$ (resp. $\triangleright_{\text{STLC}}$) means $\vdash_{\text{Curry}}$ or $\vdash_{\text{Church}}$ (resp. $\triangleright_{\text{Curry}}$ or $\triangleright_{\text{Church}}$) indifferently.

The simply typed λ -calculus in Church-style

Idea. Let us make types intrinsic to terms (static typing, *a priori*): Church-style.

$\rightsquigarrow$ Every abstracted variable in a term is associated with some type.

(λ -)Terms: $s, t ::= x \mid \lambda x^A.t \mid st$ (where A is any type, as defined for Curry-style)

Environments and judgments are as in Curry-style. The typing rules of Church-style are:

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Derivability of a judgment and typability of a λ -term are defined as in Curry-style.

β -reduction ($t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t):

$$(\lambda x^A.t)s \rightarrow_{\beta} t\{s/x\}$$

Rmk. Syntax-directedness and proposition on p. 21 hold true in Church-style as well.

Notations. $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$ means that $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in Curry/Church-style.

$\Gamma \vdash_{\text{Curry/Church}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$.

$\vdash_{\text{STLC}}$ (resp. $\triangleright_{\text{STLC}}$) means $\vdash_{\text{Curry}}$ or $\vdash_{\text{Church}}$ (resp. $\triangleright_{\text{Curry}}$ or $\triangleright_{\text{Church}}$) indifferently.

The simply typed λ -calculus in Church-style

Idea. Let us make types intrinsic to terms (static typing, *a priori*): Church-style.

$\rightsquigarrow$ Every abstracted variable in a term is associated with some type.

(λ -)Terms: $s, t ::= x \mid \lambda x^A.t \mid st$ (where A is any type, as defined for Curry-style)

Environments and judgments are as in Curry-style. The typing rules of Church-style are:

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Derivability of a judgment and typability of a λ -term are defined as in Curry-style.

β -reduction ($t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t):

$$(\lambda x^A.t)s \rightarrow_{\beta} t\{s/x\}$$

Rmk. Syntax-directedness and proposition on p. 21 hold true in Church-style as well.

Notations. $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$ means that $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in Curry/Church-style.

$\Gamma \vdash_{\text{Curry/Church}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$.

$\vdash_{\text{STLC}}$ (resp. $\triangleright_{\text{STLC}}$) means $\vdash_{\text{Curry}}$ or $\vdash_{\text{Church}}$ (resp. $\triangleright_{\text{Curry}}$ or $\triangleright_{\text{Church}}$) indifferently.

The simply typed λ -calculus in Church-style

Idea. Let us make types intrinsic to terms (static typing, *a priori*): Church-style.

↪ Every abstracted variable in a term is associated with some type.

(λ -)Terms: $s, t ::= x \mid \lambda x^A.t \mid st$ (where A is any type, as defined for Curry-style)

Environments and judgments are as in Curry-style. The typing rules of Church-style are:

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Derivability of a judgment and typability of a λ -term are defined as in Curry-style.

β -reduction ($t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t):

$$(\lambda x^A.t)s \rightarrow_{\beta} t\{s/x\}$$

Rmk. Syntax-directedness and proposition on p. 21 hold true in Church-style as well.

Notations. $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$ means that $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in Curry/Church-style.

$\Gamma \vdash_{\text{Curry/Church}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$.

$\vdash_{\text{STLC}}$ (resp. $\triangleright_{\text{STLC}}$) means $\vdash_{\text{Curry}}$ or $\vdash_{\text{Church}}$ (resp. $\triangleright_{\text{Curry}}$ or $\triangleright_{\text{Church}}$) indifferently.

The simply typed λ -calculus in Church-style

Idea. Let us make types intrinsic to terms (static typing, *a priori*): Church-style.

$\rightsquigarrow$ Every abstracted variable in a term is associated with some type.

(λ -)Terms: $s, t ::= x \mid \lambda x^A.t \mid st$ (where A is any type, as defined for Curry-style)

Environments and judgments are as in Curry-style. The typing rules of Church-style are:

$$\frac{}{\Gamma, x : A \vdash x : A} \text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.t : A \Rightarrow B} \lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A} @$$

Derivability of a judgment and typability of a λ -term are defined as in Curry-style.

β -reduction ($t\{s/x\}$ is the capture-avoiding substitution of s for the free occurrences of x in t):

$$(\lambda x^A.t)s \rightarrow_{\beta} t\{s/x\}$$

Rmk. Syntax-directedness and proposition on p. 21 hold true in Church-style as well.

Notations. $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$ means that $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash t : A$ in Curry/Church-style.

$\Gamma \vdash_{\text{Curry/Church}} t : A$ means that there is a derivation $\mathcal{D} \triangleright_{\text{Curry/Church}} \Gamma \vdash t : A$.

$\vdash_{\text{STLC}}$ (resp. $\triangleright_{\text{STLC}}$) means $\vdash_{\text{Curry}}$ or $\vdash_{\text{Church}}$ (resp. $\triangleright_{\text{Curry}}$ or $\triangleright_{\text{Church}}$) indifferently.

Curry-style versus Church-style

Church-style terms are related to Curry-style terms by the **forgetful** function $[\cdot]$:

$$[x] = x \quad [\lambda x^A.t] = \lambda x.[t] \quad [st] = [s][t]$$

Proposition

- 1 If $\Gamma \vdash_{\text{Church}} t : A$, then $\Gamma \vdash_{\text{Curry}} [t] : A$.
- 2 If $\Gamma \vdash_{\text{Curry}} t : A$ then $\Gamma \vdash_{\text{Church}} t' : A$ for some t' in Church-style such that $[t'] = t$.

Proof. By induction on the derivation in Church (Point 1) or Curry (Point 2) style. $\square$

Rmk: $\lambda x^X.x$ and $\lambda x^{X \Rightarrow X}.x$ are different terms in Church-style, because $X \neq X \Rightarrow X$.

Proposition (Uniqueness of type and derivation for typable terms in Church-style)

If $\mathcal{D} \triangleright_{\text{Church}} \Gamma \vdash t : A$ and $\mathcal{D}' \triangleright_{\text{Church}} \Gamma \vdash t : A'$, then $A = A'$ and $\mathcal{D} = \mathcal{D}'$.

Proof. By structural induction on t (exercise!). $\square$

$\rightsquigarrow$ A **bijection** between typable terms in Church-style and derivations in ND/ND_{seq}.

$\rightsquigarrow$ As β -reduction and cut-elimination mimic each other, it is an **isomorphism**.

Curry-style versus Church-style

Church-style terms are related to Curry-style terms by the **forgetful** function $[\cdot]$:

$$[x] = x \quad [\lambda x^A.t] = \lambda x.[t] \quad [st] = [s][t]$$

Proposition

- 1 If $\Gamma \vdash_{\text{Church}} t : A$, then $\Gamma \vdash_{\text{Curry}} [t] : A$.
- 2 If $\Gamma \vdash_{\text{Curry}} t : A$ then $\Gamma \vdash_{\text{Church}} t' : A$ for some t' in Church-style such that $[t'] = t$.

Proof. By induction on the derivation in Church (Point 1) or Curry (Point 2) style. $\square$

Rmk: $\lambda x^X.x$ and $\lambda x^{X \Rightarrow X}.x$ are different terms in Church-style, because $X \neq X \Rightarrow X$.

Proposition (Uniqueness of type and derivation for typable terms in Church-style)

If $\mathcal{D} \triangleright_{\text{Church}} \Gamma \vdash t : A$ and $\mathcal{D}' \triangleright_{\text{Church}} \Gamma \vdash t : A'$, then $A = A'$ and $\mathcal{D} = \mathcal{D}'$.

Proof. By structural induction on t (exercise!). $\square$

$\rightsquigarrow$ A **bijection** between typable terms in Church-style and derivations in ND/ND_{seq}.

$\rightsquigarrow$ As β -reduction and cut-elimination mimic each other, it is an **isomorphism**.

Curry-style versus Church-style

Church-style terms are related to Curry-style terms by the **forgetful** function $[\cdot]$:

$$[x] = x \quad [\lambda x^A.t] = \lambda x.[t] \quad [st] = [s][t]$$

Proposition

- 1 If $\Gamma \vdash_{\text{Church}} t : A$, then $\Gamma \vdash_{\text{Curry}} [t] : A$.
- 2 If $\Gamma \vdash_{\text{Curry}} t : A$ then $\Gamma \vdash_{\text{Church}} t' : A$ for some t' in Church-style such that $[t'] = t$.

Proof. By induction on the derivation in Church (Point 1) or Curry (Point 2) style. $\square$

Rmk: $\lambda x^X.x$ and $\lambda x^{X \Rightarrow X}.x$ are different terms in Church-style, because $X \neq X \Rightarrow X$.

Proposition (Uniqueness of type and derivation for typable terms in Church-style)

If $\mathcal{D} \triangleright_{\text{Church}} \Gamma \vdash t : A$ and $\mathcal{D}' \triangleright_{\text{Church}} \Gamma \vdash t : A'$, then $A = A'$ and $\mathcal{D} = \mathcal{D}'$.

Proof. By structural induction on t (exercise!). $\square$

$\rightsquigarrow$ A **bijection** between typable terms in Church-style and derivations in ND/ND_{seq}.

$\rightsquigarrow$ As β -reduction and cut-elimination mimic each other, it is an **isomorphism**.

Some properties of the simply typed λ -calculus (Curry and Church style)

Lemma (Substitution)

If $\Gamma, x : B \vdash_{\text{STLC}} t : A$ and $\Gamma \vdash_{\text{STLC}} s : B$, then $\Gamma \vdash_{\text{STLC}} t\{s/x\} : A$.

Proof. By structural induction on t (exercise!). □

Theorem (Subject reduction)

If $\Gamma \vdash_{\text{STLC}} t : A$ and $t \rightarrow_{\beta} s$, then $\Gamma \vdash_{\text{STLC}} s : A$.

Proof. By structural induction on t , using the substitution lemma in the key-case. □

Rmk. The converse (subject expansion) does not hold: let $A = Y \Rightarrow X \Rightarrow X$, $s = \lambda y. \lambda z. z$ and $t = (\lambda x. \lambda y. \lambda z. (xz)(yz)) \lambda x. \lambda y. x$, then $t \rightarrow_{\beta}^* s$ and $\vdash_{\text{STLC}} s : A$, but $\not\vdash_{\text{STLC}} t : A$.

Theorem (Normalization)

If $\Gamma \vdash_{\text{STLC}} t : A$, then there is $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash s : A$ without redexes such that $t \rightarrow_{\beta}^* s$.

Proof. Exactly the proof of cut-elimination in ND (uppermost in ND $\rightsquigarrow$ innermost in λ). □

Some properties of the simply typed λ -calculus (Curry and Church style)

Lemma (Substitution)

If $\Gamma, x : B \vdash_{\text{STLC}} t : A$ and $\Gamma \vdash_{\text{STLC}} s : B$, then $\Gamma \vdash_{\text{STLC}} t\{s/x\} : A$.

Proof. By structural induction on t (exercise!). □

Theorem (Subject reduction)

If $\Gamma \vdash_{\text{STLC}} t : A$ and $t \rightarrow_{\beta} s$, then $\Gamma \vdash_{\text{STLC}} s : A$.

Proof. By structural induction on t , using the substitution lemma in the key-case. □

Rmk. The converse (subject expansion) does not hold: let $A = Y \Rightarrow X \Rightarrow X$, $s = \lambda y. \lambda z. z$ and $t = (\lambda x. \lambda y. \lambda z. (xz)(yz)) \lambda x. \lambda y. x$, then $t \rightarrow_{\beta}^* s$ and $\vdash_{\text{STLC}} s : A$, but $\not\vdash_{\text{STLC}} t : A$.

Theorem (Normalization)

If $\Gamma \vdash_{\text{STLC}} t : A$, then there is $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash s : A$ without redexes such that $t \rightarrow_{\beta}^* s$.

Proof. Exactly the proof of cut-elimination in ND (uppermost in ND $\rightsquigarrow$ innermost in λ). □

Some properties of the simply typed λ -calculus (Curry and Church style)

Lemma (Substitution)

If $\Gamma, x : B \vdash_{\text{STLC}} t : A$ and $\Gamma \vdash_{\text{STLC}} s : B$, then $\Gamma \vdash_{\text{STLC}} t\{s/x\} : A$.

Proof. By structural induction on t (exercise!). □

Theorem (Subject reduction)

If $\Gamma \vdash_{\text{STLC}} t : A$ and $t \rightarrow_{\beta} s$, then $\Gamma \vdash_{\text{STLC}} s : A$.

Proof. By structural induction on t , using the substitution lemma in the key-case. □

Rmk. The converse (subject expansion) does not hold: let $A = Y \Rightarrow X \Rightarrow X$, $s = \lambda y. \lambda z. z$ and $t = (\lambda x. \lambda y. \lambda z. (xz)(yz)) \lambda x. \lambda y. x$, then $t \rightarrow_{\beta}^* s$ and $\vdash_{\text{STLC}} s : A$, but $\not\vdash_{\text{STLC}} t : A$.

Theorem (Normalization)

If $\Gamma \vdash_{\text{STLC}} t : A$, then there is $\mathcal{D} \triangleright_{\text{STLC}} \Gamma \vdash s : A$ without redexes such that $t \rightarrow_{\beta}^* s$.

Proof. Exactly the proof of cut-elimination in ND (uppermost in ND $\rightsquigarrow$ innermost in λ). □

The Curry-Howard correspondence in a nutshell

minimal logic	simply typed λ -calculus	computer science
formula	type	specification
derivation	term	program
cut-elimination step	β -reduction	computation step
derivation without redexes	normal form	result
cut-elimination theorem	normalization	termination

Concerning the correspondence between derivations and terms:

derivation in minimal logic	term in simply typed λ -calculus
hypotheses	variable
$\Rightarrow_i$	abstraction λ
$\Rightarrow_e$	application $@$

The Curry-Howard correspondence in a nutshell

minimal logic	simply typed λ -calculus	computer science
formula	type	specification
derivation	term	program
cut-elimination step	β -reduction	computation step
derivation without redexes	normal form	result
cut-elimination theorem	normalization	termination

Concerning the correspondence between derivations and terms:

derivation in minimal logic	term in simply typed λ -calculus
hypotheses	variable
$\Rightarrow_i$	abstraction λ
$\Rightarrow_e$	application $@$

Outline

- 1 Natural deduction for minimal logic, revisited
- 2 The Curry-Howard correspondence, informally
- 3 The simply typed λ -calculus
- 4 Conclusion, exercises and bibliography

What have we learned today?

- ➊ How to decorate derivations in natural deduction for minimal logic with λ -terms.
- ➋ The procedure of β -reduction on λ -terms.
- ➌ Church and Curry styles for the simply typed λ -calculus (STLC).
- ➍ The Curry–Howard correspondence between ND for minimal logic and STLC.
- ➎ Some properties of the simply typed λ -calculus (subject reduction, normalization).

Questions?

What have we learned today?

- ➊ How to decorate derivations in natural deduction for minimal logic with λ -terms.
- ➋ The procedure of β -reduction on λ -terms.
- ➌ Church and Curry styles for the simply typed λ -calculus (STLC).
- ➍ The Curry–Howard correspondence between ND for minimal logic and STLC.
- ➎ Some properties of the simply typed λ -calculus (subject reduction, normalization).

Questions?

What have we learned today?

- ➊ How to decorate derivations in natural deduction for minimal logic with λ -terms.
- ➋ The procedure of β -reduction on λ -terms.
- ➌ Church and Curry styles for the simply typed λ -calculus (STLC).
- ➍ The Curry–Howard correspondence between ND for minimal logic and STLC.
- ➎ Some properties of the simply typed λ -calculus (subject reduction, normalization).

Questions?

What have we learned today?

- 1 How to decorate derivations in natural deduction for minimal logic with λ -terms.
- 2 The procedure of β -reduction on λ -terms.
- 3 Church and Curry styles for the simply typed λ -calculus (STLC).
- 4 The Curry–Howard correspondence between ND for minimal logic and STLC.
- 5 Some properties of the simply typed λ -calculus (subject reduction, normalization).

Questions?



What have we learned today?

- 1 How to decorate derivations in natural deduction for minimal logic with λ -terms.
- 2 The procedure of β -reduction on λ -terms.
- 3 Church and Curry styles for the simply typed λ -calculus (STLC).
- 4 The Curry–Howard correspondence between ND for minimal logic and STLC.
- 5 Some properties of the simply typed λ -calculus (subject reduction, normalization).

Questions?



Non-assessed exercises (assessed exercises are available on: [click here](#))

- 1 Find the simply typed λ -terms (in Curry-style and Church-style) associated with the derivations in ND found for the facts below (see Exercise 1 from Lecture 1).
 - 1 $\vdash X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)$.
 - 2 $((X \Rightarrow Y) \Rightarrow X \Rightarrow Z)^z \vdash Y \Rightarrow X \Rightarrow Z$.
 - 3 $((X \Rightarrow Y) \Rightarrow X)^z \vdash Y \Rightarrow X$.
 - 4 $(X \Rightarrow Y \Rightarrow Z)^z \vdash Y \Rightarrow X \Rightarrow Z$.
 - 5 $(X \Rightarrow Y \Rightarrow Z)^z, (X \Rightarrow Y)^y \vdash X \Rightarrow Z$.
 - 6 $((X \Rightarrow X) \Rightarrow Y)^y \vdash (Y \Rightarrow Z) \Rightarrow Z$.
- 2 Perform all possible β -reduction steps from the λ -term decorating the derivation $\mathcal{D}$ in ND on p. 31 of Lecture 1, until you get a β -normal form. Is it always the same? Compare it with the normal derivation obtained by cut-elimination steps from $\mathcal{D}$.
- 3 Prove rigorously the following facts ($f^n x = \overbrace{f(\dots(f x) \dots)}^{n \text{ times } f}$ for any $n \in \mathbb{N}$):
 - 1 $\lambda x.xx$ is untypable in Curry-style, $\lambda x^A.xx$ is untypable in Church-style for any type A ;
 - 2 in Church-style, $\lambda f^Y.\lambda x^X.f^n x$ is not typable for any $n > 0$ but $\lambda f^Y.\lambda x^X.x$ is typable;
 - 3 $\lambda f.\lambda x.f^n x$ is typable in Curry-style, for all $n \in \mathbb{N}$.
- 4 Prove that if t is typable in Church or Curry style, then so is every subterm of t .
- 5 Prove rigorously the propositions on p. 21 and 25, the lemma and theorems on p. 26.
- 6 Let $A\{B/X\}$ be the type obtained from the type A by substituting B for each occurrence of the ground type X . Let $\Gamma\{A/X\}$ be its generalization to environments. Show that if $\Gamma \vdash_{\text{Curry}} t : A$, then $\Gamma\{B/X\} \vdash_{\text{Curry}} t : A\{B/X\}$.
- 7 Is the previous point valid in Church-style? What change is needed to make it true?
- 8 Prove that $(\lambda x.(\lambda y.y))\lambda z.zz$ is not typable (in Curry-style). Deduce that subject expansion (see p. 26) does not hold in the simply typed λ -calculus.

Bibliography

- For more about the simply typed λ -calculus:

-  Ralph Loader. *Notes on Simply Typed λ -Calculus*. Technical report ECS-LFCS-98-381, University of Edinburgh, 1998. <http://www.lfcs.inf.ed.ac.uk/reports/98/ECS-LFCS-98-381/ECS-LFCS-98-381.pdf>. [Chapters 1–3]
-  Henk Barendregt. *Lambda Calculi with Types*. In S. Abramsky et al. (eds), *Handbook of Logic in Computer Science*; vol. 2, 117-309, 1992. <https://repository.ubn.ru.nl/bitstream/handle/2066/17231/17231.pdf> [Chapter 3]

- For more about the Curry-Howard correspondence:

-  Jean-Yves Girard, Yves Lafont, Paul Taylor. *Proofs and Types*. Cambridge Tracts in Theoretical Computer Science, Vol. 7, Cambridge University Press, 1989. <https://www.paultaylor.eu/stable/prot.pdf>. [Chapters 2–4, 6]