universität innsbruck

26W lecture 7



Computability Theory

Aart Middeldorp

Outline

1. **Summary of Previous Lecture**
2. **Combinatory Logic**
3. **Combinators**
4. **Church Numerals**
5. **Combinatorial Completeness**
6. **CL–Representability**
7. **Confluence**
8. **Summary**

Definitions

- set $A \subseteq \mathbb{N}$ is recursive if its characteristic function χ_A is recursive
- disjoint sets $A, B \subseteq \mathbb{N}$ are recursively separable if there exists $f : \mathbb{N} \to \{0, 1\} \in \mathsf{R}$ such that

$$x \in A \implies f(x) = 0 \qquad x \in B \implies f(x) = 1$$

- set $A \subseteq \mathbb{N}$ is recursively enumerable if $A = \varnothing$ or A is range of unary recursive function
- set $A \subseteq \mathbb{N}$ is index set if $d \in A \wedge \varphi_e \simeq \varphi_d \implies e \in A$ for all $d, e \in \mathbb{N}$

Lemmata

- if A and B are recursively inseparable then A and B are not recursive
- set A is recursive if and only if A and $\mathbb{N} \setminus A$ are recursively enumerable

Rice's Theorem

non–trivial index sets are not recursive

Theorem

- sets $A = \{x \mid \varphi_x(x) = 0\}$ and $B = \{x \mid \varphi_x(x) = 1\}$ are recursively inseparable
- following statements are equivalent for any set $A \subseteq \mathbb{N}$:
  1. A is recursively enumerable
  2. A is range of unary partial recursive function
  3. A is domain of unary partial recursive function

Definition

set $A \subseteq \mathbb{N}$ is diophantine if $\exists$ polynomial $P(x, y_1, \ldots, y_n)$ with integer coefficients such that

$$x \in A \iff \exists\, y_1 \cdots \exists\, y_n \quad P(x, y_1, \ldots, y_n) = 0$$

Lemma

A is diophantine $\iff$ $A = \{P(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{N} \text{ and } P(x_1, \ldots, x_n) \geqslant 0\}$
for some polynomial $P(x_1, \ldots, x_n)$ with integer coefficients

Lemma

diophantine sets are recursively enumerable

Theorem (Matiyasevich)

recursively enumerable sets are diophantine

Corollary (MRDP Theorem)

Hilbert's 10th problem is unsolvable

Theorem (Jones 1975)

- $P(x, y) = 2x + 2y^3x^2 + y^2x^3 - 2yx^4 - x^5 - y^4x$ generates set of all Fibonacci numbers
- there exists no polynomial $Q(x_1, \ldots, x_n)$ such that

$$\{Q(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \geqslant 0\}$$

is set of all Fibonacci numbers

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course–of–values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s–m–n theorem, total recursive functions, undecidability, while programs

Part II: Combinatory Logic

abstraction, arithmetization, CL–representability, combinators, combinatorial completeness, Church numerals, Church–Rosser theorem, Curry–Howard isomorphism, fixed point theorem, intuitionistic propositional logic, normalization theorem, termination, typing, undecidability, Z property

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Examples

$y \qquad \mathsf{S} \qquad (\mathsf{I} \cdot \mathsf{K}) \qquad (x \cdot (\mathsf{S} \cdot y)) \qquad ((\mathsf{S} \cdot \mathsf{I}) \cdot \mathsf{I})$

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Notational Conventions

- juxtaposition for application

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Examples

y	S	$(\mathsf{I} \cdot \mathsf{K})$	$(x \cdot (\mathsf{S} \cdot y))$	$((\mathsf{S} \cdot \mathsf{I}) \cdot \mathsf{I})$	functional notation
y	S	IK	$x(\mathsf{S}y)$	SII	applicative notation

Notational Conventions

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Definitions

- set $\mathcal{V}\mathrm{ar}(t)$ of variables of CL–term t is inductively defined:

$$\mathcal{V}\mathrm{ar}(t) = \begin{cases} \varnothing & \text{if } t \in \{\mathsf{I}, \mathsf{K}, \mathsf{S}\} \\ \{t\} & \text{if } t \in \mathcal{V} \\ \mathcal{V}\mathrm{ar}(t_1) \cup \mathcal{V}\mathrm{ar}(t_2) & \text{if } t = t_1 t_2 \end{cases}$$

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$$\mathsf{B}\,xyz \to \mathsf{KS}x(\mathsf{K}x)yz \to \mathsf{S}(\mathsf{K}x)yz \to \mathsf{K}xz(yz) \to x(yz)$$

$$\mathsf{C}\,xyz \to \mathsf{BBS}x(\mathsf{KK}x)yz \to \mathsf{BBS}\,x\mathsf{K}yz \to^+ \mathsf{B}(\mathsf{S}x)\mathsf{K}yz \to^+ \mathsf{S}x(\mathsf{K}y)z \to xz(\mathsf{K}yz) \to xzy$$

Definition

$$\mathsf{B} = \mathsf{S}(\mathsf{KS})\mathsf{K} \qquad \mathsf{C} = \mathsf{S}(\mathsf{BBS})(\mathsf{KK}) \qquad \mathsf{Y} = \mathsf{B}(\mathsf{SI})(\mathsf{SII})(\mathsf{B}(\mathsf{SI})(\mathsf{SII}))$$

Lemma

$$\mathsf{B}\,xyz \;\to^+\; x(yz) \qquad \mathsf{C}\,xyz \;\to^+\; xzy \qquad \mathsf{Y}\,x \;\to^+\; x(\mathsf{Y}\,x)$$

Proof

$$\mathsf{B}\,xyz \to \mathsf{KS}x(\mathsf{K}x)yz \to \mathsf{S}(\mathsf{K}x)yz \to \mathsf{K}xz(yz) \to x(yz)$$

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Lemma

$$\mathsf{B}\,xyz \;\to^+ x(yz) \qquad \mathsf{C}\,xyz \;\to^+ xzy \qquad \mathsf{Y}\,x \;\to^+ x(\mathsf{Y}\,x)$$

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$$\mathsf{Y}\,x \to^+ \mathsf{SI}(\mathsf{SII}(\mathsf{B}(\mathsf{SI})(\mathsf{SII})))x \to \mathsf{I}\,x(\mathsf{SII}(\mathsf{B}(\mathsf{SI})(\mathsf{SII}))x)$$

Definition

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Lemma

$$\mathsf{B}\,xyz \;\to^+ x(yz) \qquad \mathsf{C}\,xyz \;\to^+ xzy \qquad \mathsf{Y}\,x \;\to^+ x(\mathsf{Y}\,x)$$

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$$\to^+ x(\mathsf{B}(\mathsf{SI})(\mathsf{SII})(\mathsf{B}(\mathsf{SI})(\mathsf{SII}))x) = x(\mathsf{Y}x)$$

Outline

1. Summary of Previous Lecture
2. Combinatory Logic
3. Combinators
4. **Church Numerals**
5. Combinatorial Completeness
6. CL–Representability
7. Confluence
8. Summary

Definition

for all $n \geqslant 0$ Church numeral $\underline{n}$ is combinator

$$\underline{n} = \begin{cases} \mathsf{KI} & \text{if } n = 0 \\ \mathsf{SB}\,\underline{n-1} & \text{if } n > 0 \end{cases}$$

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Definition

function $f\colon \mathbb{N}^n \to \mathbb{N}$ is CL−representable if there exists combinator F such that

$$f(x_1, \ldots, x_n) = y \qquad \Longrightarrow \qquad F\,\underline{x_1} \cdots \underline{x_n} \to^* \underline{y}$$

for all $x_1, \ldots, x_n, y \in \mathbb{N}$

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for all $x_1, \ldots, x_n, y \in \mathbb{N}$

$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Lemma

$\underline{n}\,x\,y \to^+ x^n\,y$

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Lemma

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Proof

induction on $n \geqslant 0$

$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

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$\underline{n}\,x\,y \to^+ x^n\,y$

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$$\underline{0}\,x\,y = \mathsf{KI}\,x\,y \to \mathsf{I}\,y \to y = x^0\,y$$

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$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Lemma

initial functions are CL−representable

$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Lemma

initial functions are CL−representable

Proof

$$\text{zero} = \mathsf{K}(\mathsf{KI})$$

$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Lemma

initial functions are CL−representable

Proof

$\mathsf{zero} = \mathsf{K}(\mathsf{KI})$:

$\mathsf{zero}\ \underline{x} \to \mathsf{KI} = \underline{0}$

$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Lemma

initial functions are CL−representable

Proof

$\mathsf{zero} = \mathsf{K}(\mathsf{KI})$:

$\mathsf{succ} = \mathsf{SB}$

$\mathsf{zero}\ \underline{x} \to \mathsf{KI} = \underline{0}$

$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Lemma

initial functions are CL−representable

Proof

$\mathsf{zero} = \mathsf{K}(\mathsf{KI})$:

$\mathsf{zero}\ \underline{x} \to \mathsf{KI} = \underline{0}$

$\mathsf{succ} = \mathsf{SB}$:

$\mathsf{succ}\ \underline{x} = \mathsf{SB}\,\underline{x} = \underline{x+1}$

$$\underline{n} = (\mathsf{SB})^n(\mathsf{KI})$$

Lemma

initial functions are CL−representable

Proof

$$\mathsf{zero} = \mathsf{K}(\mathsf{KI}) : \qquad \mathsf{zero}\ \underline{x} \to \mathsf{KI} = \underline{0}$$

$$\mathsf{succ} = \mathsf{SB} : \qquad \mathsf{succ}\ \underline{x} = \mathsf{SB}\,\underline{x} = \underline{x+1}$$

$$\pi_i^n = \begin{cases} \mathsf{K}^{i-1}((\mathsf{BK})^{n-i-1}\,\mathsf{K}) & \text{if } 1 \leqslant i < n \\ \mathsf{K}^{i-1}\,\mathsf{I} & \text{if } 1 \leqslant i = n \end{cases}$$

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Definition (Bracket Abstraction)

- CL–term $[x]t$ is defined for all CL–terms t and variables x:

$$[x]t = \begin{cases} \mathsf{I} & \text{if } t = x \\ \mathsf{K}\,t & \text{if } x \notin \mathcal{V}\mathrm{ar}(t) \\ \mathsf{S}\,([x]t_1)([x]t_2) & \text{if } t = t_1 t_2 \text{ and } x \in \mathcal{V}\mathrm{ar}(t) \end{cases}$$

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- $[x_1 \ldots x_n]t$ abbreviates $[x_1](\ldots [x_n]t \ldots)$

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Example

$[x\,y\,z](xzy)$

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Example

$$\begin{aligned}[xyz](xzy) &= [xy](\mathsf{S}([z](xz))([z]y)) = [xy](\mathsf{S}(\mathsf{S}([z]x)([z]z))(\mathsf{K}y)) = [xy](\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})(\mathsf{K}y)) \\ &= [x](\mathsf{S}([y](\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})))([y](\mathsf{K}y)))\end{aligned}$$

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Example

$$\begin{aligned}
[x\,y\,z](xzy) &= [x\,y](\mathsf{S}([z](xz))([z]y)) = [x\,y](\mathsf{S}(\mathsf{S}([z]x)([z]z))(\mathsf{K}y)) = [x\,y](\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})(\mathsf{K}y)) \\
&= [x](\mathsf{S}([y](\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})))([y](\mathsf{K}y))) = [x](\mathsf{S}(\mathsf{K}(\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})))(\mathsf{S}([y]\mathsf{K})([y]y))) \\
&= [x](\mathsf{S}(\mathsf{K}(\mathsf{S}(\mathsf{S}(\mathsf{K}x)\mathsf{I})))(\mathsf{S}(\mathsf{K}\mathsf{K})\mathsf{I})) = \cdots \\
&= \mathsf{S}(\mathsf{S}(\mathsf{K}\mathsf{S})(\mathsf{S}(\mathsf{K}\mathsf{K})(\mathsf{S}(\mathsf{K}\mathsf{S})(\mathsf{S}(\mathsf{S}(\mathsf{K}\mathsf{S})(\mathsf{S}(\mathsf{K}\mathsf{K})\mathsf{I}))(\mathsf{K}\mathsf{I})))))(\mathsf{K}(\mathsf{S}(\mathsf{K}\mathsf{K})\mathsf{I}))
\end{aligned}$$

Lemma

$x \notin \mathcal{V}\mathrm{ar}([x]t)$

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$([x]t)\,x \;\to^*\; t$ for all CL–terms t and variables x

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Proof

induction on t

- $t = x \qquad \Longrightarrow\; ([x]t)\,x = \mathsf{I}x \to x = t$

Lemma

$x \notin \mathcal{V}\mathrm{ar}([x]t)$

Lemma

$([x]t)\,x \to^* t$ for all CL–terms t and variables x

Proof

induction on t

- $t = x \qquad \Longrightarrow ([x]t)\,x = \mathsf{I}x \to x = t$
- $x \notin \mathcal{V}\mathrm{ar}(t) \qquad \Longrightarrow ([x]t)\,x = \mathsf{K}tx \to t$

Lemma

$x \notin \mathcal{V}\mathrm{ar}([x]t)$

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$([x]t)\,x \to^* t$ for all CL–terms t and variables x

Proof

induction on t

- $t = x \quad \Longrightarrow \quad ([x]t)\,x = \mathsf{I}x \to x = t$
- $x \notin \mathcal{V}\mathrm{ar}(t) \quad \Longrightarrow \quad ([x]t)\,x = \mathsf{K}tx \to t$
- $t = t_1 t_2$ and $x \in \mathcal{V}\mathrm{ar}(t) \quad \Longrightarrow \quad ([x]t_1)x \to^* t_1$ and $([x]t_2)x \to^* t_2$ by induction hypothesis

Lemma

$x \notin \mathcal{V}\mathrm{ar}([x]t)$

Lemma

$([x]t)\,x \to^* t$ for all CL–terms t and variables x

Proof

induction on t

- $t = x \quad \Longrightarrow \quad ([x]t)\,x = \mathsf{I}x \to x = t$
- $x \notin \mathcal{V}\mathrm{ar}(t) \quad \Longrightarrow \quad ([x]t)\,x = \mathsf{K}tx \to t$
- $t = t_1 t_2$ and $x \in \mathcal{V}\mathrm{ar}(t) \quad \Longrightarrow \quad ([x]t_1)x \to^* t_1$ and $([x]t_2)x \to^* t_2$ by induction hypothesis

  $\Longrightarrow \quad ([x]t)\,x = \mathsf{S}([x]t_1)([x]t_2)x \to ([x]t_1)x(([x]t_2)x) \to^* t_1 t_2 = t$

Corollary (Combinatorial Completeness)

for every CL–term t with $\mathcal{V}\mathrm{ar}(t) = \{x_1, \ldots, x_n\}$

1. $\exists$ combinator C such that $C\, x_1 \cdots x_n \to^* t$

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① $C = [x_1 \ldots x_n]t$

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Proof

① $C = [x_1 \ldots x_n]t$: $\quad C\, x_1 \cdots x_n \to^+ ([x_2 \ldots x_n]t)\, x_2 \cdots x_n \to^+ \cdots \to^+ t$

Corollary (Combinatorial Completeness)

for every CL–term t with $\mathcal{V}\mathrm{ar}(t) = \{x_1, \ldots, x_n\}$

1. $\exists$ combinator C such that $C\, x_1 \cdots x_n \to^* t$
2. $\exists$ combinator D such that $D\, x_2 \cdots x_n \to^* t\,[D/x_1]$

Proof

① $C = [x_1 \ldots x_n]t$: $\quad C\, x_1 \cdots x_n \to^+ ([x_2 \ldots x_n]t)\, x_2 \cdots x_n \to^+ \cdots \to^+ t$

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② $D = \mathsf{Y}C$

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Definition (Bracket Abstraction, Optimized)

- CL–term $\langle x \rangle t$ is defined for all CL–terms t and variables x:

$$\langle x \rangle t = \begin{cases} \mathsf{I} & \text{if } t = x \\ \mathsf{K}\, t & \text{if } x \notin \mathcal{V}\mathrm{ar}(t) \end{cases} \qquad \langle x \rangle t = \begin{cases} u & \text{if } t = u\, x \text{ and } x \notin \mathcal{V}\mathrm{ar}(u) \\ \mathsf{B}\, u\, (\langle x \rangle v) & \text{if } t = u\, v \text{ and } x \notin \mathcal{V}\mathrm{ar}(u) \\ \mathsf{C}\, (\langle x \rangle u)\, v & \text{if } t = u\, v \text{ and } x \notin \mathcal{V}\mathrm{ar}(v) \\ \mathsf{S}\, (\langle x \rangle u)\, (\langle x \rangle v) & \text{if } t = u\, v \text{ and } x \in \mathcal{V}\mathrm{ar}(u) \cap \mathcal{V}\mathrm{ar}(v) \end{cases}$$

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Example

$$\langle x\, y\, z \rangle (x z y)$$

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$$\langle x\, y\, z \rangle (x z y) = \langle x\, y \rangle (\mathsf{C} (\langle z \rangle (x z)) y)$$

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Example

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Lemma

$(\langle x \rangle t)\, x \to^* t$ for all CL–terms t and variables x

Outline

1. Summary of Previous Lecture
2. Combinatory Logic
3. Combinators
4. Church Numerals
5. Combinatorial Completeness
6. **CL–Representability**
7. Confluence
8. Summary

Lemma

CL−representable functions are closed under composition

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CL−representable functions are closed under composition

Proof

$$f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n))$$

with $G, H_1, \ldots, H_m$ representing $g, h_1, \ldots, h_m$

$$F = \langle x_1 \ldots x_n \rangle (G\,(H_1\,x_1 \cdots x_n) \cdots (H_m\,x_1 \cdots x_n))$$

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Lemma

CL−representable functions are closed under primitive recursion

$$f(0, y_1, \ldots, y_n) = g(y_1, \ldots, y_n)$$
$$f(x+1, y_1, \ldots, y_n) = h(f(x, y_1, \ldots, y_n), x, y_1, \ldots, y_n)$$

$$f(x, y_1, \ldots, y_n) = \begin{cases} g(y_1, \ldots, y_n) & \text{if } x = 0 \\ h(f(x-1, y_1, \ldots, y_n), x-1, y_1, \ldots, y_n) & \text{otherwise} \end{cases}$$

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Booleans

$\mathsf{T} = \mathsf{K}$ $\mathsf{F} = \mathsf{KI}$

$$f(x, y_1, \ldots, y_n) = \begin{cases} g(y_1, \ldots, y_n) & \text{if } x = 0 \\ h(f(x-1, y_1, \ldots, y_n), x-1, y_1, \ldots, y_n) & \text{otherwise} \end{cases}$$

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$\mathsf{T} = \mathsf{K}$ $\mathsf{F} = \mathsf{KI}$ $\mathsf{T}xy \to x$ $\mathsf{F}xy \to \mathsf{I}y \to y$

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$\mathsf{T} = \mathsf{K}$ $\mathsf{F} = \mathsf{KI}$ $\mathsf{T}xy \to x$ $\mathsf{F}xy \to \mathsf{I}y \to y$

Test for Zero

$$\mathsf{zero?} = \mathsf{C}(\mathsf{B}(\mathsf{CIK}))(\mathsf{K}(\mathsf{KI}))$$

$$f(x, y_1, \ldots, y_n) = \begin{cases} g(y_1, \ldots, y_n) & \text{if } x = 0 \\ h(f(x-1, y_1, \ldots, y_n), x-1, y_1, \ldots, y_n) & \text{otherwise} \end{cases}$$

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$\mathsf{T} = \mathsf{K}$ $\quad$ $\mathsf{F} = \mathsf{KI}$ $\quad$ $\mathsf{T}xy \to x$ $\quad$ $\mathsf{F}xy \to \mathsf{I}y \to y$

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$$\text{zero?} = \mathsf{C}(\mathsf{B}(\mathsf{CIK}))(\mathsf{K}(\mathsf{KI}))$$

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$\mathsf{T} = \mathsf{K}$ $\quad$ $\mathsf{F} = \mathsf{KI}$ $\quad$ $\mathsf{T}xy \to x$ $\quad$ $\mathsf{F}xy \to \mathsf{I}y \to y$

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$\mathsf{T} = \mathsf{K}$ $\quad$ $\mathsf{F} = \mathsf{KI}$ $\quad$ $\mathsf{T}xy \to x$ $\quad$ $\mathsf{F}xy \to \mathsf{I}y \to y$

Test for Zero

$$\mathsf{zero?} = \mathsf{C}(\mathsf{B}(\mathsf{CIK}))(\mathsf{K}(\mathsf{KI}))$$

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Booleans

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Test for Zero

$$\mathsf{zero?} = \mathsf{C}(\mathsf{B}(\mathsf{CIK}))(\mathsf{K}(\mathsf{KI}))$$

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$$\mathsf{zero?}\ \underline{n+1} \to^* \mathsf{B}(\mathsf{CIK})\,\underline{n+1}\,(\mathsf{K}(\mathsf{KI}))$$

$$f(x, y_1, \ldots, y_n) = \begin{cases} g(y_1, \ldots, y_n) & \text{if } x = 0 \\ h(f(x-1, y_1, \ldots, y_n), x-1, y_1, \ldots, y_n) & \text{otherwise} \end{cases}$$

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$\mathsf{T} = \mathsf{K}$ $\quad$ $\mathsf{F} = \mathsf{KI}$ $\quad$ $\mathsf{T}xy \to x$ $\quad$ $\mathsf{F}xy \to \mathsf{I}y \to y$

Test for Zero

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$$\mathsf{zero?}\ \underline{n+1} \to^* \mathsf{B}(\mathsf{CIK})\,\underline{n+1}\,(\mathsf{K}(\mathsf{KI})) \to^* \mathsf{CIK}(\mathsf{SB}\,\underline{n}\,(\mathsf{K}(\mathsf{KI})))$$

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Booleans

$\mathsf{T} = \mathsf{K}$ $\quad$ $\mathsf{F} = \mathsf{KI}$ $\quad$ $\mathsf{T}xy \to x$ $\quad$ $\mathsf{F}xy \to \mathsf{I}y \to y$

Test for Zero

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$$\mathsf{zero?}\ \underline{n+1} \to^* \mathsf{B}(\mathsf{CIK})\,\underline{n+1}\,(\mathsf{K}(\mathsf{KI})) \to^* \mathsf{CIK}(\mathsf{SB}\,\underline{n}\,(\mathsf{K}(\mathsf{KI}))) \to^* \mathsf{CIK}(\mathsf{B}(\mathsf{K}(\mathsf{KI}))(\underline{n}\,(\mathsf{K}(\mathsf{KI}))))$$

$$f(x, y_1, \ldots, y_n) = \begin{cases} g(y_1, \ldots, y_n) & \text{if } x = 0 \\ h(f(x-1, y_1, \ldots, y_n), x-1, y_1, \ldots, y_n) & \text{otherwise} \end{cases}$$

Booleans

$\mathsf{T} = \mathsf{K}$ $\quad$ $\mathsf{F} = \mathsf{KI}$ $\quad$ $\mathsf{T}xy \to x$ $\quad$ $\mathsf{F}xy \to \mathsf{I}y \to y$

Test for Zero

$$\mathsf{zero?} = \mathsf{C}(\mathsf{B}(\mathsf{CIK}))(\mathsf{K}(\mathsf{KI}))$$

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$$f(x, y_1, \ldots, y_n) = \begin{cases} g(y_1, \ldots, y_n) & \text{if } x = 0 \\ h(f(x-1, y_1, \ldots, y_n), x-1, y_1, \ldots, y_n) & \text{otherwise} \end{cases}$$

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Predecessor

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Lemma

P represents predecessor function

Lemma

CL−representable functions are closed under primitive recursion

Lemma

CL−representable functions are closed under primitive recursion

Proof

$$f(0, y_1, \ldots, y_n) = g(y_1, \ldots, y_n)$$
$$f(x+1, y_1, \ldots, y_n) = h(f(x, y_1, \ldots, y_n), x, y_1, \ldots, y_n)$$

with G, H representing g, h

Lemma

CL−representable functions are closed under primitive recursion

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Corollary

primitive recursive functions are CL−representable

Outline

1. Summary of Previous Lecture
2. Combinatory Logic
3. Combinators
4. Church Numerals
5. Combinatorial Completeness
6. CL–Representability
7. **Confluence**
8. Summary

Definition

function $f\colon \mathbb{N}^n \to \mathbb{N}$ is CL–representable if there exists combinator F such that

$$f(x_1, \ldots, x_n) = y \quad \Longrightarrow \quad F\,\underline{x_1} \cdots \underline{x_n} \to^* \underline{y}$$

$$f(x_1, \ldots, x_n) \text{ is undefined} \quad \Longrightarrow \quad F\,\underline{x_1} \cdots \underline{x_n} \text{ is not normalizing}$$

for all $x_1, \ldots, x_n, y \in \mathbb{N}$

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Question

is it possible that both $F\,\underline{x_1} \cdots \underline{x_n} \to^* \underline{y}$ and $F\,\underline{x_1} \cdots \underline{x_n} \to^* \underline{z}$ with $y \neq z$?

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Answer

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Answer

no, because CL has unique normal forms: $\forall$ term t $\forall$ normal forms n_1 and n_2

$$t \to^* n_1 \quad \wedge \quad t \to^* n_2 \quad \Longrightarrow \quad n_1 = n_2$$

Theorem

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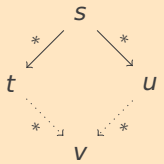

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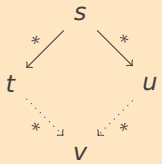

Corollary

CL has unique normal forms

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1. Summary of Previous Lecture
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8. **Summary**

Important Concepts

- $\rightarrow$
- $\rightarrow^+$
- $\rightarrow^*$
- $\rightarrow^!$
- B
- $[x]t$
- $\langle x \rangle t$
- bracket abstraction
- C
- Church numeral
- CL–representable function
- CL–term
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homework for December 7