universität innsbruck

26W lecture 6



Computability Theory

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Outline

1. **Summary of Previous Lecture**
2. **Recursive and Recursive Enumerable Sets**
3. **Diophantine Sets**
4. **Fibonacci Numbers**
5. **Summary**

Definition

$\varphi_e^n(x_1, \ldots, x_n) = \mathrm{u}((\mu\, y)\, (\mathrm{t}_n(e, x_1, \ldots, x_n, y) = 0))$

Lemma

φ_0^n, φ_1^n, φ_2^n, ... is computable enumeration of all n–ary partial recursive functions

Definition

predicate $P\colon \mathbb{N}^n \to \mathbb{B}$ is decidable if χ_P is recursive

Theorem

following problem is undecidable:

instance: natural number x

question: is $\varphi_x(x)$ defined ?

Kleene's s–m–n or Parameterization Theorem

$\forall\, m, n \geqslant 1 \quad \exists$ primitive recursive function $s_n^m \colon \mathbb{N}^{m+1} \to \mathbb{N} \quad \forall\, e \in \mathbb{N}$

$$\varphi_e^{m+n}(x_1, \ldots, x_m, y_1, \ldots, y_n) \simeq \varphi_{s_n^m(e, x_1, \ldots, x_m)}^n(y_1, \ldots, y_n)$$

Kleene's Fixed Point Theorem

$\forall$ recursive function $f \colon \mathbb{N} \to \mathbb{N} \quad \exists\, e \in \mathbb{N}$ such that $\varphi_e(x) \simeq \varphi_{f(e)}(x)$

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course–of–values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s–m–n theorem, total recursive functions, undecidability, while programs

Part II: Combinatory Logic

abstraction, arithmetization, CL–representability, combinators, combinatorial completeness, Church numerals, Church–Rosser theorem, Curry–Howard isomorphism, fixed point theorem, intuitionistic propositional logic, normalization theorem, termination, typing, undecidability, Z property

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- set $A \subseteq \mathbb{N}$ is recursive if its characteristic function $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ is recursive

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- disjoint sets $A, B \subseteq \mathbb{N}$ are recursively separable if $\exists$ recursive function $f\colon \mathbb{N} \to \{0, 1\}$ such that

$$x \in A \implies f(x) = 0 \qquad x \in B \implies f(x) = 1$$

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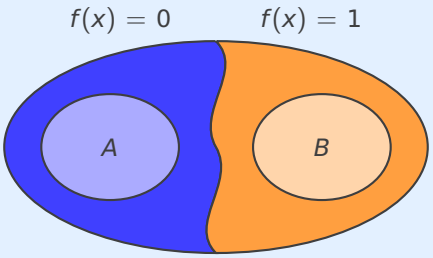

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set $A \subseteq \mathbb{N}$ is index set if

$$d \in A \wedge \varphi_e \simeq \varphi_d \quad \Longrightarrow \quad e \in A$$

for all $d, e \in \mathbb{N}$

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- $\varnothing$ and $\mathbb{N}$ are (trivial) index sets

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Rice's Theorem

non–trivial index sets are not recursive

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- let A be non–trivial index set

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- let A be non–trivial index set and let $d \in A$ and $e \notin A$

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- let A be non–trivial index set and let $d \in A$ and $e \notin A$
- suppose A is recursive

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- let A be non–trivial index set and let $d \in A$ and $e \notin A$
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set $A \subseteq \mathbb{N}$ is recursively enumerable if $A = \varnothing$ or A is range of unary recursive function

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other terminology: semi–decidable computably enumerable

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Lemma

set A is recursive if and only if A and $\mathbb{N} \setminus A$ are recursively enumerable

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other terminology: semi−decidable computably enumerable

Lemma

set A is recursive if and only if A and $\mathbb{N} \setminus A$ are recursively enumerable

Theorem

following statements are equivalent for any set $A \subseteq \mathbb{N}$:

1. A is recursively enumerable
2. A is range of unary partial recursive function
3. A is domain of unary partial recursive function

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Proof 1 $\Longrightarrow$ 3

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Proof 1 $\Longrightarrow$ 3

- $A = \varnothing$ or A is range of unary recursive function
- $\varnothing$ is domain of unary partial recursive function $f(x) = (\mu\, y)\,(x + 1 = 0)$

1. A is recursively enumerable
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Proof 1 $\Longrightarrow$ 3

- $A = \varnothing$ or A is range of unary recursive function f
- $\varnothing$ is domain of unary partial recursive function $f(x) = (\mu\, y)\,(x + 1 = 0)$
- define unary function $g(x) = (\mu\, y)\,(f(y) = x)$

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Proof 1 $\Longrightarrow$ 3

- $A = \varnothing$ or A is range of unary recursive function f
- $\varnothing$ is domain of unary partial recursive function $f(x) = (\mu\, y)\,(x + 1 = 0)$
- define unary function $g(x) = (\mu\, y)\,(f(y) = x)$
- g is partial recursive

1. A is recursively enumerable
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Proof (1 $\Longrightarrow$ 3)

- $A = \varnothing$ or A is range of unary recursive function f
- $\varnothing$ is domain of unary partial recursive function $f(x) = (\mu\, y)\,(x + 1 = 0)$
- define unary function $g(x) = (\mu\, y)\,(f(y) = x)$
- g is partial recursive
- domain of g is A

1. A is recursively enumerable
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Proof 3 ⟹ 2

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Proof 3 $\Longrightarrow$ 2

- suppose A is domain of unary partial recursive function φ_e

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Proof 3 $\Longrightarrow$ 2

- suppose A is domain of unary partial recursive function φ_e
- define unary function $f(x) = x + \mathsf{z}(\varphi_e(x))$

1. A is recursively enumerable
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Proof 3 $\Longrightarrow$ 2

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- define unary function $f(x) = x + \mathrm{z}(\varphi_e(x))$
- f is partial recursive

1. A is recursively enumerable
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Proof 3 $\Longrightarrow$ 2

- suppose A is domain of unary partial recursive function φ_e
- define unary function $f(x) = x + \mathsf{z}(\varphi_e(x))$
- f is partial recursive
- $\varphi_e(x)\downarrow \iff f(x)\downarrow$

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- range of f is A

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Proof 2 $\Longrightarrow$ 1

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- if $A = \varnothing$ then A is recursively enumerable

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Proof 2 $\Longrightarrow$ 1

- suppose A is range of unary partial recursive function φ_e
- if $A = \varnothing$ then A is recursively enumerable suppose $A \neq \varnothing$ and let $a \in A$

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Proof (2 $\Longrightarrow$ 1)

- suppose A is range of unary partial recursive function φ_e
- if $A = \varnothing$ then A is recursively enumerable suppose $A \neq \varnothing$ and let $a \in A$
- define $f(x,s) = \begin{cases} \varphi_e(x) & \text{if } (\exists\, y < s)\, \mathsf{T}_1(e,x,y) \\ a & \text{otherwise} \end{cases}$ and $g(x) = f(\pi_1(x), \pi_2(x))$

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- $z \in A \implies z = \varphi_e(x)$ for some x

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- claim: range of g is A
- $z \in A \implies z = \varphi_e(x)$ for some $x \implies \mathsf{T}_1(e,x,y)$ for some y
- $f(x,s) = z$ for any $s > y$

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Proof 2 $\Longrightarrow$ 1

- suppose A is range of unary partial recursive function φ_e
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- f and g are recursive
- claim: range of g is A
- $z \in A \implies z = \varphi_e(x)$ for some $x \implies \mathrm{T}_1(e,x,y)$ for some y
- $f(x,s) = z$ for any $s > y \implies g(\pi(x,s)) = z$ for any $s > y$

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every non−empty recursively enumerable set $A \subseteq \mathbb{N}$ is range of primitive recursive function

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- A is range of unary recursive function f

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$$g(x) = \begin{cases} \mathrm{u}((x)_1) & \text{if } \mathrm{T}_1(e, (x)_0, (x)_1) \\ a & \text{otherwise} \end{cases}$$

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Outline

1. Summary of Previous Lecture
2. Recursive and Recursive Enumerable Sets
3. **Diophantine Sets**
4. Fibonacci Numbers
5. Summary

Definition

set $A \subseteq \mathbb{N}$ is diophantine if $\exists$ polynomial $P(x, y_1, \ldots, y_n)$ with integer coefficients such that

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Corollary (MRDP Theorem)

Hilbert's 10th problem is unsolvable

Breaking News (OpenAI, 24 September 2026)

Hilbert's 10th problem over $\mathbb{Q}$ is unsolvable

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Example

polynomial $P(a, b, \ldots, z)$:

$$
\begin{aligned}
(k+2)\Big(1 &- (wz+h+j-q)^2 - \big((gk+2g+k+1)(h+j)+h-z\big)^2 \\
&- (2n+p+q+z-e)^2 - \big((a^2-1)y^2+1-x^2\big)^2 \\
&- \big(16(k+1)^3(k+2)(n+1)^2+1-f^2\big)^2 - (n+l+v-y)^2 \\
&- \big(16r^2y^4(a^2-1)+1-u^2\big)^2 - (e^3(e+2)(a+1)^2+1-o^2)^2 \\
&- \big(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2\big)^2 \\
&- \big((a^2-1)l^2+1-m^2\big)^2 - (ai+k+1-l-i)^2 \\
&- \big(p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m\big)^2 \\
&- \big(z+pl(a-p)+t(2ap-p^2-1)-pm\big)^2 \\
&- \big(q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x\big)^2\Big)
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Example (Jones, Sato, Wada, Wiens 1976)

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generates all prime numbers

Examples (Prime Generating Functions)

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2. Recursive and Recursive Enumerable Sets
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Definition (Fibonacci numbers)

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Theorem (Jones 1975)

$P(x, y) = 2x + 2y^3x^2 + y^2x^3 - 2yx^4 - x^5 - y^4x$ generates set of Fibonacci numbers

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induction on x

- $x = 0 \implies y = 1$
- $x > 0$

Lemma ❷

$y^2 - yx - x^2 = 1 \wedge x, y \geqslant 0 \quad \Longrightarrow \quad x = F_{2i}$ and $y = F_{2i+1}$ for some $i \geqslant 0$

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- $x = 0 \quad \Longrightarrow \quad y = 1 \quad \Longrightarrow \quad i = 0$
- $x > 0 \quad \Longrightarrow \quad yx + x^2 > y$

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Proof

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- $x > 0 \implies yx + x^2 > y \implies 1 = y^2 - (yx + x^2) < y^2 - y$

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Lemma ❸

$y^2 - yx - x^2 = -1 \;\wedge\; x \geqslant 0 \;\wedge\; y > 0 \quad \Longrightarrow \quad x = F_{2i+1}$ and $y = F_{2i+2}$ for some $i \geqslant 0$

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Proof

case analysis

- $x \leqslant y$

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Corollary 1

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Corollary

set of Fibonacci numbers is diophantine

Lemma 4

$$y > 0 \wedge x \geqslant 0 \quad \Longrightarrow \quad y^2 - yx - x^2 \neq 0$$

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- $x = 0 \quad \Longrightarrow \quad y^2 - yx - x^2 = y^2 > 0$
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$$4(y^2 - yx - x^2) = (2y - x)^2 - 5x^2$$

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$$(2y - x)^2 - 5x^2 = 0 \quad \Longrightarrow \quad \sqrt{5} = \frac{|2y - x|}{x}$$



Lemma 4

$y > 0 \wedge x \geqslant 0 \quad \Longrightarrow \quad y^2 - yx - x^2 \neq 0$

Proof

case analysis

- $x = 0 \quad \Longrightarrow \quad y^2 - yx - x^2 = y^2 > 0$
- $x > 0$

$4(y^2 - yx - x^2) = (2y - x)^2 - 5x^2$

$(2y - x)^2 - 5x^2 = 0 \quad \Longrightarrow \quad \sqrt{5} = \dfrac{|2y - x|}{x}$ ↯

$(2y - x)^2 - 5x^2 \neq 0$

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$$(2y - x)^2 - 5x^2 \neq 0 \quad \Longrightarrow \quad y^2 - yx - x^2 \neq 0$$

Theorem

set of Fibonacci numbers equals non−negative values of $P(x, y) = x(2 - (y^2 - yx - x^2)^2)$

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two directions

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  - $y = 0$
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  - $y = 0 \implies z = x(2-x^4)$
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    $(y^2-yx-x^2)^2 = 1 \implies z = x$

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- x is Fibonacci number $\implies$ $x = F_i$ for some $i \geqslant 0$

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  - $y > 0 \implies y^2-yx-x^2 \neq 0$ by lemma 4 $\implies 0 < (y^2-yx-x^2)^2 < 2$

    $(y^2-yx-x^2)^2 = 1 \implies z = x \implies z$ is Fibonacci number by corollary 2

Theorem (Jones 1975)

$P(x, y) = 2x + 2y^3x^2 + y^2x^3 - 2yx^4 - x^5 - y^4x$ generates set of Fibonacci numbers

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Remark

$P(2,2) = -28$

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Remark

$P(2, 2) = -28$

Theorem

there exists no polynomial $Q(x_1, \ldots, x_n)$ such that

$$\{Q(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \geqslant 0\}$$

is set of all Fibonacci numbers

Outline

1. Summary of Previous Lecture
2. Recursive and Recursive Enumerable Sets
3. Diophantine Sets
4. Fibonacci Numbers
5. **Summary**

Important Concepts

- diophantine set
- Fibonacci numbers
- index set
- recursive set
- recursive enumerable set
- recursive separable sets

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- diophantine set
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homework for November 30