universität innsbruck

26W lecture 5



Computability Theory

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Outline

1. **Summary of Previous Lecture**
2. **Normal Form Theorem**
3. **Undecidability**
4. **s–m–n or Parameterization Theorem**
5. **Fixed Point Theorem**
6. **Summary**

Theorem

class R of recursive functions is smallest class of total functions that contains all projection functions, addition and multiplication, characteristic function $\chi_=$ of equality predicate, and is closed under composition and minimization

Definition

$\beta(a,i) = \pi_1(a) \bmod \big(1 + (i+1)\,\pi_2(a)\big)$ Gödel's β function

Definition

class PA of partial recursive functions is smallest class of partial functions that contains all initial functions and is closed under composition, primitive recursion and unbounded minimization:

$$(\mu\, i)\,(f(i,\vec{y}) = 0) = \min\{i \mid f(i,\vec{y}) = 0 \text{ and } f(j,\vec{y}) > 0 \text{ for all } j < i\}$$

belongs to PA whenever $f\colon \mathbb{N}^{n+1} \to \mathbb{N}$ belongs to PA

Definition

function $f\colon \mathbb{N}^n \to \mathbb{N}$ is while computable if $\exists$ while program $P(x_1, \ldots, x_n; y)$ such that $y = f(x_1, \ldots, x_n)$ after execution of P

Theorem

function φ is partial recursive $\iff$ φ is while computable

Definition

index $\ulcorner f \urcorner \in \mathbb{N}$ of derivation of partial recursive function f is defined inductively:

- $\ulcorner z \urcorner = \langle 0 \rangle$ $\ulcorner s \urcorner = \langle 1 \rangle$ $\ulcorner \pi_i^n \urcorner = \langle 2, n, i \rangle$
- $\ulcorner f \urcorner = \langle 3, \ulcorner g \urcorner, \ulcorner h_1 \urcorner, \ldots, \ulcorner h_m \urcorner \rangle$ if f is obtained by composing g and $h_1, \ldots, h_m$
- $\ulcorner f \urcorner = \langle 4, \ulcorner g \urcorner, \ulcorner h \urcorner \rangle$ if f is obtained by primitive recursion from g and h
- $\ulcorner f \urcorner = \langle 5, \ulcorner g \urcorner \rangle$ if f is obtained by minimizing g

Kleene's Normal Form Theorem

$\exists$ primitive recursive function u $\quad \forall\, n \geqslant 1 \quad \exists$ primitive recursive predicate T_n
$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N} \qquad \varphi(x_1, \ldots, x_n) \simeq \mathsf{u}((\mu\, y)\, \mathsf{T}_n(\ulcorner\varphi\urcorner, x_1, \ldots, x_n, y))$

Corollary

partial recursiveness and recursiveness coincide for total functions

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course–of–values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s–m–n theorem, total recursive functions, undecidability, while programs

Part II: Combinatory Logic

abstraction, arithmetization, CL–representability, combinators, combinatorial completeness, Church numerals, Church–Rosser theorem, Curry–Howard isomorphism, fixed point theorem, intuitionistic propositional logic, normalization theorem, termination, typing, undecidability, Z property

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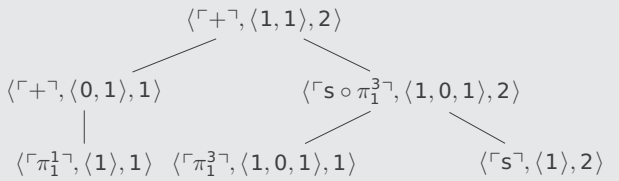
Example

$$0 + y = \pi_1^1(y)$$
$$(x+1) + y = \mathsf{s}(\pi_1^3(x + y, x, y))$$

$1 + 1 = 2$:



$\langle\ulcorner f \urcorner, \langle x_1, \ldots, x_n\rangle, z\rangle$ represents $f(x_1, \ldots, x_n) = z$

Notation

$\mathrm{triple}(x) = (\mathrm{seq}(x) \wedge \mathrm{len}(x) = 3)$

Key Idea

- encode computation tree $T = \langle x, y, z \rangle$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \ldots, \ulcorner T_n \urcorner \rangle$



- dependencies between node $\langle x, y, z \rangle$ and root nodes of its children $T_1, \ldots, T_n$ can be checked by primitive recursive predicate D

Definitions

- $\mathsf{T}(x) \iff \mathsf{D}(x) \wedge \big[\, \mathrm{len}(x) > 1 \iff (\forall\, i \leqslant \mathrm{len}(x) \dot{-} 2)\, \mathsf{T}((x)_{i+2}) \,\big]$
- $\mathsf{T}_n(e, \vec{x}, y) \iff \mathsf{T}(y) \wedge (y)_{1,1} = e \wedge (y)_{1,2} = \langle \vec{x} \rangle$
- $\mathsf{u}(x) = (x)_{1,3}$

Theorem

$\varphi(x_1, \ldots, x_n) \simeq \mathsf{u}((\mu\, y)\, \mathsf{T}_n(\ulcorner \varphi \urcorner, x_1, \ldots, x_n, y))$

Definition of D (cont'd)

- zero $\langle \langle \langle 0 \rangle, \langle x \rangle, 0 \rangle \rangle$
- successor $\langle \langle \langle 1 \rangle, \langle x \rangle, x + 1 \rangle \rangle$
- projection $\langle \langle \langle 2, n, i \rangle, \langle x_1, \ldots, x_n \rangle, x_i \rangle \rangle$

$$\mathsf{initial}(x) \iff \mathrm{seq}(x) \wedge \mathrm{len}(x) = 1 \wedge \mathrm{triple}((x)_1) \wedge \mathrm{seq}((x)_{1,1}) \wedge \mathrm{seq}((x)_{1,2}) \wedge \big[\, \mathsf{zero}((x)_1) \vee \mathsf{successor}((x)_1) \vee \mathsf{projection}((x)_1) \,\big]$$

with

$$\mathsf{zero}(x) \iff (x)_1 = \langle 0 \rangle \wedge \mathrm{len}((x)_2) = 1 \wedge (x)_3 = 0$$

$$\mathsf{successor}(x) \iff (x)_1 = \langle 1 \rangle \wedge \mathrm{len}((x)_2) = 1 \wedge (x)_3 = \mathrm{s}((x)_{2,1})$$

$$\mathsf{projection}(x) \iff \mathrm{len}((x)_1) = 3 \wedge (x)_{1,1} = 2 \wedge (x)_{1,2} = \mathrm{len}((x)_2) \wedge (x)_{1,3} \geqslant 1 \wedge (x)_{1,2} \geqslant (x)_{1,3} \wedge (x)_3 = (x)_{2,(x)_{1,3}}$$

Definition of D (cont'd)

- composition $f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n))$

$$\langle \langle \ulcorner f \urcorner, \langle x_1, \ldots, x_n \rangle, z \rangle, T_1, \ldots, T_m, T_{m+1} \rangle$$

with

$$\ulcorner f \urcorner = \langle 3, \ulcorner g \urcorner, \ulcorner h_1 \urcorner, \ldots, \ulcorner h_m \urcorner \rangle$$
$$T_1 = \langle \langle \ulcorner h_1 \urcorner, \langle x_1, \ldots, x_n \rangle, z_1 \rangle, \ldots \rangle \quad \cdots \quad T_m = \langle \langle \ulcorner h_m \urcorner, \langle x_1, \ldots, x_n \rangle, z_m \rangle, \ldots \rangle$$
$$T_{m+1} = \langle \langle \ulcorner g \urcorner, \langle z_1, \ldots, z_m \rangle, z \rangle, \ldots \rangle$$

$$\begin{aligned} \mathsf{composition}(x) \iff\ & \mathrm{seq}(x) \wedge \mathrm{len}(x) \geqslant 3 \wedge \mathrm{triple}((x)_1) \wedge \mathrm{seq}((x)_{1,1}) \wedge (x)_{1,1,1} = 3 \wedge \\ & \mathrm{len}((x)_{1,1}) = \mathrm{len}(x) \wedge (\forall\, i \leqslant \mathrm{len}(x) \dot{-} 3)\, \big[(x)_{i+2,1,1} = (x)_{1,1,i+3} \wedge \\ & (x)_{i+2,1,2} = (x)_{1,2} \wedge (x)_{i+2,1,3} = (x)_{\mathrm{len}(x),1,2,i+1} \big] \wedge \\ & (x)_{\mathrm{len}(x),1,1} = (x)_{1,1,2} \wedge (x)_{\mathrm{len}(x),1,3} = (x)_{1,3} \end{aligned}$$

Definition of D (cont'd)

- primitive recursion $f(0, \vec{y}) = g(\vec{y})$ $f(x+1, \vec{y}) = h(f(x, \vec{y}), x, \vec{y})$

$$\langle \langle \ulcorner f \urcorner, \langle 0, y_1, \ldots, y_n \rangle, z \rangle, \langle \langle \ulcorner g \urcorner, \langle y_1, \ldots, y_n \rangle, z \rangle, \ldots \rangle \rangle \quad \langle \langle \ulcorner f \urcorner, \langle x+1, y_1, \ldots, y_n \rangle, z \rangle, T_1, T_2 \rangle$$

with

$$\ulcorner f \urcorner = \langle 4, \ulcorner g \urcorner, \ulcorner h \urcorner \rangle$$
$$T_1 = \langle \langle \ulcorner f \urcorner, \langle x, y_1, \ldots, y_n \rangle, z_1 \rangle, \cdots \rangle \qquad T_2 = \langle \langle \ulcorner h \urcorner, \langle z_1, x, y_1, \ldots, y_n \rangle, z \rangle, \cdots \rangle$$

$$\begin{aligned} \mathsf{recursion}(x) \iff\ & \mathrm{seq}(x) \wedge \mathrm{triple}((x)_1) \wedge \mathrm{triple}((x)_{1,1}) \wedge (x)_{1,1,1} = 4 \wedge \mathrm{seq}((x)_{1,2}) \wedge \\ & \big[(x)_{1,2,1} = 0 \implies \mathrm{len}(x) = 2 \wedge (x)_{1,1,2} = (x)_{2,1,1} \wedge \\ & (x)_{1,2} = \langle 0 \rangle\ ;\ (x)_{2,1,2} \wedge (x)_{1,3} = (x)_{2,1,3} \big] \wedge \big[(x)_{1,2,1} > 0 \implies \\ & \mathrm{len}(x) = 3 \wedge (x)_{2,1,1} = (x)_{1,1} \wedge (x)_{2,1,2,1} = \mathrm{p}((x)_{1,2,1}) \wedge \\ & (\forall\, i \leqslant \mathrm{len}((x)_{1,2}) \dot{-} 2)\, \big[(x)_{1,2,i+2} = (x)_{2,1,2,i+2} \big] \wedge (x)_{3,1,1} = (x)_{1,1,3} \wedge \\ & (x)_{3,1,2} = \langle (x)_{2,1,3} \rangle\ ;\ (x)_{2,1,2} \wedge (x)_{3,1,3} = (x)_{1,3} \big] \end{aligned}$$

Definition of D (cont'd)

- minimization $f(y_1, \ldots, y_n) = (\mu x)\,(g(x, y_1, \ldots, y_n) = 0)$

  minimization(x) $\iff \cdots$ (homework exercise)

- $\mathrm{D}(x) \iff$ initial(x) $\vee$ composition(x) $\vee$ recursion(x) $\vee$ minimization(x)

Kleene's Normal Form Theorem

$\exists$ primitive recursive function u $\quad \forall\, n \geqslant 1 \quad \exists$ primitive recursive predicate T_n
$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$

$$\varphi(x_1, \ldots, x_n) \simeq \mathrm{u}((\mu\, y)\, \mathsf{T}_n(\ulcorner\varphi\urcorner, x_1, \ldots, x_n, y))$$

Corollary

$\exists$ primitive recursive function u $\quad \forall\, n \geqslant 1 \quad \exists$ primitive recursive function t_n
$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N} \qquad \varphi(x_1, \ldots, x_n) \simeq \mathrm{u}((\mu\, y)\, (\mathrm{t}_n(\ulcorner\varphi\urcorner, x_1, \ldots, x_n, y) = 0))$

Proof

define $\mathrm{t}_n = \chi_{\neg \mathsf{T}_n}$

Definition

$\varphi_e^n(x_1, \ldots, x_n) = \mathrm{u}((\mu\, y)\, (\mathrm{t}_n(e, x_1, \ldots, x_n, y) = 0))$

Lemma

- $\forall\, e \geqslant 0 \quad \forall\, n \geqslant 1 \quad \varphi_e^n$ is partial recursive
- e is no index $\implies$ φ_e^n is nowhere defined
- $\varphi_0^n,\ \varphi_1^n,\ \varphi_2^n, \ldots$ is enumeration of all n–ary partial recursive functions

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Definition

predicate $P\colon \mathbb{N}^n \to \mathbb{B}$ is decidable if χ_P is recursive

Notation

φ_e denotes φ_e^1

Theorem

following problem is undecidable:

instance: natural number x
question: is $\varphi_x(x)$ defined ?

Proof (by contradiction)

- suppose $f(x) = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{if } \varphi_x(x)\uparrow \end{cases}$ is recursive
- function $g(x) = (\mu\, i)\,(f(x) = 0)$ is partial recursive
- $\exists\, e$ such that $g = \varphi_e$
- $g(e)\downarrow \iff f(e) = 0 \iff \varphi_e(e)\uparrow$ ↯

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Example

- $x + y$ is partial recursive $\implies$ $\exists\, e$ such that $x + y = \varphi_e^2(x, y)$
- $2 + y$ is partial recursive $\implies$ $\exists\, e'$ such that $2 + y = \varphi_{e'}(y)$
- claim: e' can be computed from e and 2 by primitive recursion

Kleene's s–m–n or Parameterization Theorem

$\forall\, m, n \geqslant 1$ $\exists$ primitive recursive function $s_n^m \colon \mathbb{N}^{m+1} \to \mathbb{N}$ $\forall\, e \in \mathbb{N}$

$$\varphi_e^{m+n}(x_1, \ldots, x_m, y_1, \ldots, y_n) \simeq \varphi^n_{s_n^m(e, x_1, \ldots, x_m)}(y_1, \ldots, y_n)$$

Proof

- primitive recursive function $f \colon \mathbb{N} \to \mathbb{N}$ computes index of n–ary constant function c_x:

$$f(0) = \langle 3, \langle 0\rangle, \langle 2, n, 1\rangle\rangle \qquad f(x+1) = \ulcorner s \circ c_x \urcorner = \langle 3, \langle 1\rangle, f(x)\rangle$$

- $s_n^m(e, x_1, \ldots, x_m) = \langle 3, e, f(x_1), \ldots, f(x_m), \langle 2, n, 1\rangle, \ldots, \langle 2, n, n\rangle\rangle$

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Kleene's Fixed Point Theorem

$\forall$ recursive function $f\colon \mathbb{N} \to \mathbb{N}$ $\exists\, e \in \mathbb{N}$ such that $\varphi_e(x) \simeq \varphi_{f(e)}(x)$

Proof

- $\psi(x,y) = \begin{cases} \varphi_{\varphi_x(x)}(y) & \text{if } \varphi_x(x)\downarrow \\ \text{undefined} & \text{otherwise} \end{cases}$ is partial recursive
- define $g(x) = s_1^1(\ulcorner\psi\urcorner, x)$
- s–m–n theorem $\implies$ $\psi(x,y) \simeq \varphi_{g(x)}(y)$
- $f \circ g$ is recursive $\implies$ $\exists\, d$ such that $f(g(x)) = \varphi_d(x)$
- take $e = g(d)$: $\varphi_e(x) = \varphi_{g(d)}(x) \simeq \psi(d,x) \simeq \varphi_{\varphi_d(d)}(x) = \varphi_{f(g(d))}(x) = \varphi_{f(e)}(x)$

Example

task: construct partial recursive function that evaluates to its own index

- π_1^2 is partial recursive $\implies$ $\exists\, n$ such that $\pi_1^2(x,y) = \varphi_n^2(x,y)$
- s–m–n theorem $\implies$ $\varphi_n^2(x,y) = \varphi_{g(x)}(y)$ for $g(x) = s_1^1(n,x)$
- fixed point theorem $\implies$ $\exists\, m$ such that $\varphi_m(x) \simeq \varphi_{g(m)}(x)$
- $\varphi_m(x) \simeq \varphi_{s_1^1(n,m)}(x) = \varphi_n^2(m,x) = \pi_1^2(m,x) = m$

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Important Concepts

- decidable predicate
- φ_e
- φ_e^n
- Kleene's fixed point theorem
- Kleene's normal form theorem
- Kleene's s–m–n theorem
- s_n^m
- T_n
- t_n
- u



homework for November 23