universität innsbruck

26W lecture 5



Computability Theory

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Outline

1. **Summary of Previous Lecture**
2. **Normal Form Theorem**
3. **Undecidability**
4. **s–m–n or Parameterization Theorem**
5. **Fixed Point Theorem**
6. **Summary**

Theorem

class R of recursive functions is smallest class of total functions that contains all projection functions, addition and multiplication, characteristic function $\chi_=$ of equality predicate, and is closed under composition and minimization

Definition

$\beta(a,i) = \pi_1(a) \bmod \big(1 + (i+1)\,\pi_2(a)\big)$ Gödel's β function

Definition

class PA of partial recursive functions is smallest class of partial functions that contains all initial functions and is closed under composition, primitive recursion and unbounded minimization:

$$(\mu\, i)\,(f(i,\vec{y}) = 0) = \min\{i \mid f(i,\vec{y}) = 0 \text{ and } f(j,\vec{y}) > 0 \text{ for all } j < i\}$$

belongs to PA whenever $f\colon \mathbb{N}^{n+1} \to \mathbb{N}$ belongs to PA

Definition

function $f\colon \mathbb{N}^n \to \mathbb{N}$ is while computable if $\exists$ while program $P(x_1, \ldots, x_n; y)$ such that $y = f(x_1, \ldots, x_n)$ after execution of P

Theorem

function φ is partial recursive $\iff$ φ is while computable

Definition

index $\ulcorner f \urcorner \in \mathbb{N}$ of derivation of partial recursive function f is defined inductively:

- $\ulcorner \mathsf{z} \urcorner = \langle 0 \rangle \qquad \ulcorner \mathsf{s} \urcorner = \langle 1 \rangle \qquad \ulcorner \pi_i^n \urcorner = \langle 2, n, i \rangle$
- $\ulcorner f \urcorner = \langle 3, \ulcorner g \urcorner, \ulcorner h_1 \urcorner, \ldots, \ulcorner h_m \urcorner \rangle$ if f is obtained by composing g and $h_1, \ldots, h_m$
- $\ulcorner f \urcorner = \langle 4, \ulcorner g \urcorner, \ulcorner h \urcorner \rangle$ if f is obtained by primitive recursion from g and h
- $\ulcorner f \urcorner = \langle 5, \ulcorner g \urcorner \rangle$ if f is obtained by minimizing g

Kleene's Normal Form Theorem

$\exists$ primitive recursive function u $\quad \forall\, n \geqslant 1$ $\quad \exists$ primitive recursive predicate T_n

$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$ $\qquad \varphi(x_1, \ldots, x_n) \simeq \mathsf{u}((\mu\, y)\, \mathsf{T}_n(\ulcorner\varphi\urcorner, x_1, \ldots, x_n, y))$

Corollary

partial recursiveness and recursiveness coincide for total functions

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course–of–values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s–m–n theorem, total recursive functions, undecidability, while programs

Part II: Combinatory Logic

abstraction, arithmetization, CL–representability, combinators, combinatorial completeness, Church numerals, Church–Rosser theorem, Curry–Howard isomorphism, fixed point theorem, intuitionistic propositional logic, normalization theorem, termination, typing, undecidability, Z property

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2. **Normal Form Theorem**

3. Undecidability

4. s–m–n or Parameterization Theorem

5. Fixed Point Theorem

6. Summary

Example

$$0 + y = \pi_1^1(y)$$
$$(x + 1) + y = \mathsf{s}(\pi_1^3(x + y, x, y))$$

$1 + 1 = 2$

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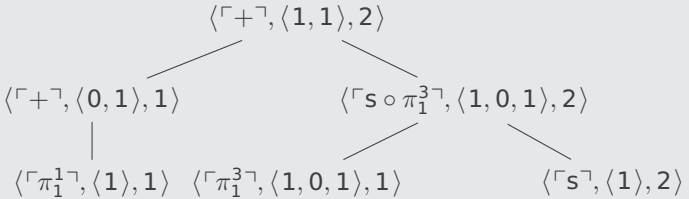

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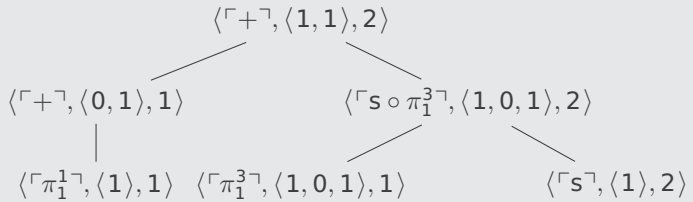

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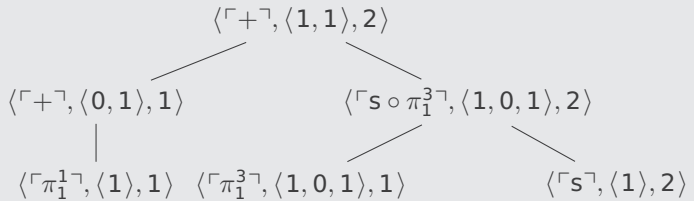

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Notation

$\mathsf{triple}(x) = (\mathsf{seq}(x) \wedge \mathsf{len}(x) = 3)$

Key Idea

- encode computation tree $T = \langle x, y, z \rangle$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \ldots, \ulcorner T_n \urcorner \rangle$



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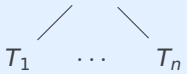

- dependencies between node $\langle x, y, z \rangle$ and root nodes of its children $T_1, \ldots, T_n$ can be checked by primitive recursive predicate

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- dependencies between node $\langle x, y, z \rangle$ and root nodes of its children $T_1, \ldots, T_n$ can be checked by primitive recursive predicate D

Definitions

- $\mathrm{T}(x) \iff \mathrm{D}(x) \wedge \big[\, \mathrm{len}(x) > 1 \iff (\forall\, i \leqslant \mathrm{len}(x) \dot{-} 2)\, \mathrm{T}((x)_{i+2}) \,\big]$

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- $\mathsf{u}(x) = (x)_{1,3}$

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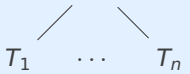

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- $\mathsf{u}(x) = (x)_{1,3}$

Theorem

$\varphi(x_1, \ldots, x_n) \simeq \mathsf{u}((\mu\, y)\, \mathsf{T}_n(\ulcorner \varphi \urcorner, x_1, \ldots, x_n, y))$

Definition of D (cont'd)

$$\text{initial}(x) \iff \text{seq}(x) \wedge \text{len}(x) = 1 \wedge \text{triple}((x)_1) \wedge \text{seq}((x)_{1,1}) \wedge \text{seq}((x)_{1,2}) \wedge \big[\, \text{zero}((x)_1) \vee \text{successor}((x)_1) \vee \text{projection}((x)_1) \,\big]$$

Definition of D (cont'd)

- zero $\langle\langle\langle 0\rangle, \langle x\rangle, 0\rangle\rangle$

$$\text{initial}(x) \iff \text{seq}(x) \wedge \text{len}(x) = 1 \wedge \text{triple}((x)_1) \wedge \text{seq}((x)_{1,1}) \wedge \text{seq}((x)_{1,2}) \wedge \big[\, \text{zero}((x)_1) \vee \text{successor}((x)_1) \vee \text{projection}((x)_1) \,\big]$$

with

$$\text{zero}(x) \iff (x)_1 = \langle 0\rangle \wedge \text{len}((x)_2) = 1 \wedge (x)_3 = 0$$

Definition of D (cont'd)

- zero $\quad \langle\langle\langle 0\rangle, \langle x\rangle, 0\rangle\rangle$
- successor $\quad \langle\langle\langle 1\rangle, \langle x\rangle, x+1\rangle\rangle$

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with

$$\text{zero}(x) \iff (x)_1 = \langle 0\rangle \wedge \text{len}((x)_2) = 1 \wedge (x)_3 = 0$$

$$\text{successor}(x) \iff (x)_1 = \langle 1\rangle \wedge \text{len}((x)_2) = 1 \wedge (x)_3 = \text{s}((x)_{2,1})$$

Definition of D (cont'd)

- zero $\langle\langle\langle 0\rangle, \langle x\rangle, 0\rangle\rangle$
- successor $\langle\langle\langle 1\rangle, \langle x\rangle, x+1\rangle\rangle$
- projection $\langle\langle\langle 2, n, i\rangle, \langle x_1, \ldots, x_n\rangle, x_i\rangle\rangle$

$$\begin{aligned}\text{initial}(x) \iff\ & \text{seq}(x) \wedge \text{len}(x) = 1 \wedge \text{triple}((x)_1) \wedge \text{seq}((x)_{1,1}) \wedge \text{seq}((x)_{1,2}) \wedge \\ & \big[\, \text{zero}((x)_1) \vee \text{successor}((x)_1) \vee \text{projection}((x)_1) \,\big]\end{aligned}$$

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$$\begin{aligned}\text{zero}(x) &\iff (x)_1 = \langle 0\rangle \wedge \text{len}((x)_2) = 1 \wedge (x)_3 = 0 \\ \text{successor}(x) &\iff (x)_1 = \langle 1\rangle \wedge \text{len}((x)_2) = 1 \wedge (x)_3 = \text{s}((x)_{2,1}) \\ \text{projection}(x) &\iff \text{len}((x)_1) = 3 \wedge (x)_{1,1} = 2 \wedge (x)_{1,2} = \text{len}((x)_2) \wedge (x)_{1,3} \geqslant 1 \wedge \\ &\quad\ (x)_{1,2} \geqslant (x)_{1,3} \wedge (x)_3 = (x)_{2,(x)_{1,3}}\end{aligned}$$

Definition of D (cont'd)

- composition $f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n))$

$$\langle\langle \ulcorner f \urcorner, \langle x_1, \ldots, x_n \rangle, z\rangle, T_1, \ldots, T_m, T_{m+1}\rangle$$

with

$$\ulcorner f \urcorner = \langle 3, \ulcorner g \urcorner, \ulcorner h_1 \urcorner, \ldots, \ulcorner h_m \urcorner\rangle$$

$$T_1 = \langle\langle \ulcorner h_1 \urcorner, \langle x_1, \ldots, x_n\rangle, z_1\rangle, \ldots\rangle \quad \cdots \quad T_m = \langle\langle \ulcorner h_m \urcorner, \langle x_1, \ldots, x_n\rangle, z_m\rangle, \ldots\rangle$$

$$T_{m+1} = \langle\langle \ulcorner g \urcorner, \langle z_1, \ldots, z_m\rangle, z\rangle, \ldots\rangle$$

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T_{m+1} &= \langle\langle \ulcorner g \urcorner, \langle z_1, \ldots, z_m\rangle, z\rangle, \ldots\rangle
\end{aligned}$$

$$\begin{aligned}
\text{composition}(x) \iff\ & \text{seq}(x) \wedge \text{len}(x) \geqslant 3 \wedge \text{triple}((x)_1) \wedge \text{seq}((x)_{1,1}) \wedge (x)_{1,1,1} = 3 \wedge \\
& \text{len}((x)_{1,1}) = \text{len}(x) \wedge (\forall\, i \leqslant \text{len}(x) \dot{-} 3)\, \big[\, (x)_{i+2,1,1} = (x)_{1,1,i+3} \wedge \\
& (x)_{i+2,1,2} = (x)_{1,2} \wedge (x)_{i+2,1,3} = (x)_{\text{len}(x),1,2,i+1} \,\big] \wedge \\
& (x)_{\text{len}(x),1,1} = (x)_{1,1,2} \wedge (x)_{\text{len}(x),1,3} = (x)_{1,3}
\end{aligned}$$

Definition of D (cont'd)

- primitive recursion $f(0,\vec{y}) = g(\vec{y})$ $\quad f(x+1,\vec{y}) = h(f(x,\vec{y}),x,\vec{y})$

$$\langle\langle\ulcorner f \urcorner, \langle 0, y_1, \ldots, y_n\rangle, z\rangle, \langle\langle\ulcorner g\urcorner, \langle y_1, \ldots, y_n\rangle, z\rangle, \ldots\rangle\rangle \quad \langle\langle\ulcorner f\urcorner, \langle x+1, y_1, \ldots, y_n\rangle, z\rangle, T_1, T_2\rangle$$

with

$$\ulcorner f \urcorner = \langle 4, \ulcorner g\urcorner, \ulcorner h\urcorner\rangle$$

$$T_1 = \langle\langle\ulcorner f\urcorner, \langle x, y_1, \ldots, y_n\rangle, z_1\rangle, \cdots\rangle \qquad T_2 = \langle\langle\ulcorner h\urcorner, \langle z_1, x, y_1, \ldots, y_n\rangle, z\rangle, \cdots\rangle$$

Definition of D (cont'd)

- primitive recursion $\quad f(0,\vec{y}) = g(\vec{y}) \qquad f(x+1,\vec{y}) = h(f(x,\vec{y}),x,\vec{y})$

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$$T_1 = \langle\langle\ulcorner f\urcorner, \langle x, y_1, \ldots, y_n\rangle, z_1\rangle, \cdots\rangle \qquad T_2 = \langle\langle\ulcorner h\urcorner, \langle z_1, x, y_1, \ldots, y_n\rangle, z\rangle, \cdots\rangle$$

$$\begin{aligned}\text{recursion}(x) \iff\ & \mathrm{seq}(x) \wedge \mathrm{triple}((x)_1) \wedge \mathrm{triple}((x)_{1,1}) \wedge (x)_{1,1,1} = 4 \wedge \mathrm{seq}((x)_{1,2}) \wedge \\ & \big[(x)_{1,2,1} = 0 \implies \mathrm{len}(x) = 2 \wedge (x)_{1,1,2} = (x)_{2,1,1} \wedge \\ & (x)_{1,2} = \langle 0\rangle\ ;\ (x)_{2,1,2} \wedge (x)_{1,3} = (x)_{2,1,3}\big] \wedge \big[(x)_{1,2,1} > 0 \implies \\ & \mathrm{len}(x) = 3 \wedge (x)_{2,1,1} = (x)_{1,1} \wedge (x)_{2,1,2,1} = \mathrm{p}((x)_{1,2,1}) \wedge \\ & (\forall\, i \leqslant \mathrm{len}((x)_{1,2}) \dot{-} 2)\ \big[(x)_{1,2,i+2} = (x)_{2,1,2,i+2}\big] \wedge (x)_{3,1,1} = (x)_{1,1,3} \wedge \\ & (x)_{3,1,2} = \langle (x)_{2,1,3}\rangle\ ;\ (x)_{2,1,2} \wedge (x)_{3,1,3} = (x)_{1,3}\big]\end{aligned}$$

Definition of D (cont'd)

- minimization $f(y_1, \ldots, y_n) = (\mu\, x)\,(g(x, y_1, \ldots, y_n) = 0)$

  minimization(x) $\iff \cdots$ (homework exercise)

Definition of D (cont'd)

- minimization $f(y_1, \ldots, y_n) = (\mu\, x)\,(g(x, y_1, \ldots, y_n) = 0)$

  $\text{minimization}(x) \iff \cdots$ (homework exercise)

- $\text{D}(x) \iff \text{initial}(x) \vee \text{composition}(x) \vee \text{recursion}(x) \vee \text{minimization}(x)$

Definition of D (cont'd)

- minimization $f(y_1, \ldots, y_n) = (\mu\, x)\,(g(x, y_1, \ldots, y_n) = 0)$

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Kleene's Normal Form Theorem

$\exists$ primitive recursive function u $\forall\, n \geqslant 1$ $\exists$ primitive recursive predicate T_n
$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$

$$\varphi(x_1, \ldots, x_n) \simeq \mathrm{u}((\mu\, y)\, \mathrm{T}_n(\ulcorner\varphi\urcorner, x_1, \ldots, x_n, y))$$

Corollary

$\exists$ primitive recursive function u $\quad \forall\, n \geqslant 1$ $\quad \exists$ primitive recursive function t_n

$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$ $\qquad \varphi(x_1, \ldots, x_n) \simeq \mathsf{u}((\mu y)\,(\mathsf{t}_n(\ulcorner\varphi\urcorner, x_1, \ldots, x_n, y) = 0))$

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Proof

define $\mathsf{t}_n = \chi_{\neg \mathsf{T}_n}$

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Definition

$\varphi_e^n(x_1, \ldots, x_n) = \mathsf{u}((\mu\, y)\, (\mathsf{t}_n(e, x_1, \ldots, x_n, y) = 0))$

Corollary

$\exists$ primitive recursive function u $\quad \forall\, n \geqslant 1 \quad \exists$ primitive recursive function t_n
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$\varphi_e^n(x_1, \ldots, x_n) = \mathsf{u}((\mu\, y)\, (\mathsf{t}_n(e, x_1, \ldots, x_n, y) = 0))$

Lemma

- $\forall\, e \geqslant 0 \quad \forall\, n \geqslant 1 \quad \varphi_e^n$ is partial recursive

Corollary

$\exists$ primitive recursive function u $\quad \forall\, n \geqslant 1 \quad \exists$ primitive recursive function t_n
$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N} \qquad \varphi(x_1, \ldots, x_n) \simeq \mathsf{u}((\mu\, y)\,(\mathsf{t}_n(\ulcorner\varphi\urcorner, x_1, \ldots, x_n, y) = 0))$

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Lemma

- $\forall\, e \geqslant 0 \quad \forall\, n \geqslant 1 \quad \varphi_e^n$ is partial recursive
- e is no index $\quad \Longrightarrow \quad \varphi_e^n$ is nowhere defined

Corollary

$\exists$ primitive recursive function u $\quad \forall\, n \geqslant 1$ $\quad \exists$ primitive recursive function t_n

$\forall$ partial recursive function $\varphi\colon \mathbb{N}^n \to \mathbb{N}$ $\qquad \varphi(x_1, \ldots, x_n) \simeq \mathsf{u}((\mu\, y)\,(\mathsf{t}_n(\ulcorner\varphi\urcorner, x_1, \ldots, x_n, y) = 0))$

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Definition

$\varphi_e^n(x_1, \ldots, x_n) = \mathsf{u}((\mu\, y)\,(\mathsf{t}_n(e, x_1, \ldots, x_n, y) = 0))$

Lemma

- $\forall\, e \geqslant 0 \quad \forall\, n \geqslant 1 \quad \varphi_e^n$ is partial recursive
- e is no index $\quad \Longrightarrow \quad \varphi_e^n$ is nowhere defined
- $\varphi_0^n,\ \varphi_1^n,\ \varphi_2^n,\ \ldots$ is enumeration of all n–ary partial recursive functions

Outline

1. Summary of Previous Lecture
2. Normal Form Theorem
3. **Undecidability**
4. s–m–n or Parameterization Theorem
5. Fixed Point Theorem
6. Summary

Definition

predicate $P\colon \mathbb{N}^n \to \mathbb{B}$ is decidable if χ_P is recursive

Definition

predicate $P\colon \mathbb{N}^n \to \mathbb{B}$ is decidable if χ_P is recursive

Notation

φ_e denotes φ_e^1

Definition

predicate $P\colon \mathbb{N}^n \to \mathbb{B}$ is decidable if χ_P is recursive

Notation

φ_e denotes φ_e^1

Theorem

following problem is undecidable:

instance: natural number x

question: is $\varphi_x(x)$ defined ?

Proof (by contradiction)

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- $g(e)\downarrow$

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Outline

1. Summary of Previous Lecture
2. Normal Form Theorem
3. Undecidability
4. **s–m–n or Parameterization Theorem**
5. Fixed Point Theorem
6. Summary

Example

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$\forall\, m, n \geqslant 1 \quad \exists$ primitive recursive function $s_n^m : \mathbb{N}^{m+1} \to \mathbb{N} \quad \forall\, e \in \mathbb{N}$

$$\varphi_e^{m+n}(x_1, \ldots, x_m, y_1, \ldots, y_n) \simeq \varphi^n_{s_n^m(e, x_1, \ldots, x_m)}(y_1, \ldots, y_n)$$

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Important Concepts

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- Kleene's fixed point theorem
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- T_n
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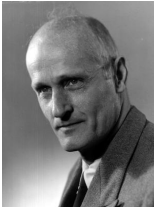

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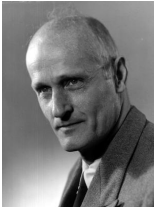

homework for November 23