



Computability Theory

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Outline

- 1. Summary of Previous Lecture**
- 2. Loop Programs**
- 3. Elementary Functions**
- 4. Grzegorzcyk Hierarchy**
- 5. Summary**

Definitions

- ▶ $\langle x_1, \dots, x_n \rangle = p_0^n \times p_1^{x_1} \times \dots \times p_n^{x_n}$ **Gödel number**
- ▶ $(x)_i = (\mu j < x) \neg (p_i^{j+1} \mid x)$
- ▶ $\text{len}(x) = (x)_0$
- ▶ $\text{seq}(x) \iff x > 0 \wedge (\forall i \leq x) [(x)_i \neq 0 \implies i \leq \text{len}(x)]$
- ▶ $x ; y = p_0^{\text{len}(x) + \text{len}(y)} \times \prod_{i < \text{len}(x)} p_{i+1}^{(x)_{i+1}} \times \prod_{i < \text{len}(y)} p_{\text{len}(x) + i + 1}^{(y)_{i+1}}$

Lemma

PR is closed under **course-of-values recursion**:

if $g: \mathbb{N}^n \rightarrow \mathbb{N}$ and $h: \mathbb{N}^{n+2} \rightarrow \mathbb{N}$ are primitive recursive then

$$f(0, \vec{y}) = g(\vec{y})$$

$$f(x+1, \vec{y}) = h(\tilde{f}(x, \vec{y}), x, \vec{y})$$

with $\tilde{f}(x, \vec{y}) = \langle f(0, \vec{y}), \dots, f(x, \vec{y}) \rangle$ is primitive recursive

Definition (Ackermann function)

$$\text{ack}(0, y) = y + 1$$

$$\text{ack}(x + 1, 0) = \text{ack}(x, 1)$$

$$\text{ack}(x + 1, y + 1) = \text{ack}(x, \text{ack}(x + 1, y))$$

Lemma

$\forall$ primitive recursive function $f: \mathbb{N}^n \rightarrow \mathbb{N}$ $\exists$ constant $c \in \mathbb{N}$ such that

$$f(x_1, \dots, x_n) < \text{ack}(c, \max \{x_1, \dots, x_n\})$$

Theorem

Ackermann function is **not** primitive recursive

Definition

class **R** of **recursive functions** is smallest class of **total** functions that contains all initial functions and is closed under composition, primitive recursion and **(unbounded) minimization**:

$$(\mu i) P(i, \vec{y}) = \min \{i \mid P(i, \vec{y})\} \in R$$

for all $P: \mathbb{N}^{n+1} \rightarrow \mathbb{B}$ such that $\chi_P \in R$ and $\forall \vec{y} \exists x P(x, \vec{y})$

Theorem

► $\exists$ primitive recursive predicate $P: \mathbb{N} \rightarrow \mathbb{B}$ such that

$$P(x) \iff x \text{ encodes correct Ackermann computation tree}$$

► $\text{ack}(x, y) = z \iff \exists t \text{ such that } P(t) \text{ and } (t)_1 = \langle x, y, z \rangle$

► $\text{ack}(x, y) = ((\mu t) (P(t) \wedge (t)_{1,1} = x \wedge (t)_{1,2} = y))_{1,3}$

► Ackermann function is recursive

Definition

index $\ulcorner f \urcorner \in \mathbb{N}$ of derivation of primitive recursive function f is defined inductively:

- ▶ $\ulcorner z \urcorner = \langle 0 \rangle$
- ▶ $\ulcorner s \urcorner = \langle 1 \rangle$
- ▶ $\ulcorner \pi_i^n \urcorner = \langle 2, n, i \rangle$
- ▶ $\ulcorner f \urcorner = \langle 3, \ulcorner g \urcorner, \ulcorner h_1 \urcorner, \dots, \ulcorner h_m \urcorner \rangle$ if f is obtained by composing g and $h_1, \dots, h_m$
- ▶ $\ulcorner f \urcorner = \langle 4, \ulcorner g \urcorner, \ulcorner h \urcorner \rangle$ if f is obtained by primitive recursion from g and h

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course-of-values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s-m-n theorem, total recursive functions, undecidability, while programs

Part II: Combinatory Logic

abstraction, arithmetization, CL-representability, combinators, combinatorial completeness, Church numerals, Church-Rosser theorem, Curry-Howard isomorphism, fixed point theorem, intuitionistic propositional logic, normalization theorem, termination, typing, undecidability, Z property

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Loop Programs

simple programming language

- ▶ natural numbers are only data type

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- ▶ variables $x, y, z, \dots$

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- ▶ commands
 - ▶ assignment $x := 0$

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 - ▶ composition $P; Q$

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 - ▶ loop
 - ▶ **LOOP x DO P OD**

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 - ▶ composition $P; Q$
 - ▶ loop
 - ▶ **LOOP x DO P OD**

execute P exactly n times, where n is value of x before entering loop

Examples

program P :

```
z := x;  
LOOP y DO  
  z++  
OD
```

Examples

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computes addition: $z = x + y$

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notation: $P(x, y; z)$

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  z++  
OD
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notation: $P(x, y; z)$

multiplication $(x, y; z)$:

```
z := 0;  
LOOP x DO  
  LOOP y DO  
    z++  
  OD  
OD
```

Examples

program $P(x; y)$:

$y := 0$;

$z := 0$;

LOOP x DO

$y := z$;

$z++$

OD

Examples

program $P(x; y)$:

$y := 0$;

$z := 0$;

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$y := z$;

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computes predecessor: $y = p(x)$

Examples

program $P(x; y)$:

```
y := 0;  
z := 0;  
LOOP x DO  
  y := z;  
  z++  
OD
```

computes predecessor: $y = p(x)$

program $Q(x; y)$:

```
y := 0;  
z := 0;  
z++;  
LOOP x DO  
  y := z  
OD
```

Examples

program $P(x; y)$:

```
y := 0;  
z := 0;  
LOOP x DO  
  y := z;  
  z++  
OD
```

computes predecessor: $y = p(x)$

program $Q(x; y)$:

```
y := 0;  
z := 0;  
z++;  
LOOP x DO  
  y := z  
OD
```

computes sign function: $y = sg(x)$

Lemma

loop programs **terminate**

Lemma

loop programs terminate

Definition

function $f: \mathbb{N}^n \rightarrow \mathbb{N}$ is **loop computable** if $\exists$ loop program $P(x_1, \dots, x_n; y)$ such that

$$y = f(x_1, \dots, x_n)$$

after execution of P

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after execution of P

Theorem

primitive recursive functions are loop computable

► zero

$y = z(x)$

$y := 0$

Proof

- ▶ zero $y = z(x)$ $y := 0$
- ▶ successor $y = s(x)$ $y := x; y++$

Proof

- | | | |
|--------------|--------------------------------|---------------|
| ▶ zero | $y = z(x)$ | $y := 0$ |
| ▶ successor | $y = s(x)$ | $y := x; y++$ |
| ▶ projection | $y = \pi_i^n(x_1, \dots, x_n)$ | $y := x_i$ |

Proof

- ▶ zero $y = z(x)$ $y := 0$
- ▶ successor $y = s(x)$ $y := x; y++$
- ▶ projection $y = \pi_i^n(x_1, \dots, x_n)$ $y := x_i$
- ▶ composition $y = g(h_1(x_1, \dots, x_m), \dots, h_n(x_1, \dots, x_m))$

$$P_{h_1}(x_1, \dots, x_m; y_1); \dots; P_{h_n}(x_1, \dots, x_m; y_n); P_g(y_1, \dots, y_n; y)$$

Proof

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$$P_{h_1}(x_1, \dots, x_m; y_1); \dots; P_{h_n}(x_1, \dots, x_m; y_n); P_g(y_1, \dots, y_n; y)$$

- ▶ primitive recursion

$$\begin{aligned} f(0, \vec{y}) &= g(\vec{y}) \\ f(x+1, \vec{y}) &= h(f(x, \vec{y}), x, \vec{y}) \end{aligned}$$

```
 $P_g(y_1, \dots, y_n; z);$   
 $v := 0;$   
LOOP  $x$  DO  
     $P_h(z, v, y_1, \dots, y_n; w);$   
     $z := w;$   
     $v++$   
OD
```

Theorem

loop computable functions are primitive recursive

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Proof

- ▶ loop program P with variables $x_1, \dots, x_n$

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- ▶ loop program P with variables $x_1, \dots, x_n$
- ▶ claim: $\exists$ primitive recursive functions $f_1, \dots, f_n: \mathbb{N}^n \rightarrow \mathbb{N}$ such that
if $x_1, \dots, x_n$ have values $a_1, \dots, a_n$ before execution of P
then x_i has value $f_i(a_1, \dots, a_n)$ after execution of P , for all $1 \leq i \leq n$

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- ▶ claim is proved by induction on structure of P (details left as exercise)

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Example

$f_P(x, y) = f_3(x, y, 0, 0, 0)$ for loop program P with input variables x_1 and x_2 ,
output variable x_3 , and auxiliary variables x_4 and x_5

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class **E** of **elementary functions** is smallest class of (total) functions $f: \mathbb{N}^n \rightarrow \mathbb{N}$ that contains all initial functions, $+$, $\cdot$ and is closed under composition, bounded summation and bounded product

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$$\triangleright x \times y = \sum_{i < x} y = \sum_{i < x} \pi_2^2(i, y)$$

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Lemma

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Proof

► consider $f(x, \vec{y}) = (\mu i \leq x) (g(i, \vec{y}) = 0)$ with $g \in E$

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- ▶ consider $f(x, \vec{y}) = (\mu i \leq x) (g(i, \vec{y}) = 0)$ with $g \in E$
- ▶ elementary function

$$f'(x, \vec{y}) = \sum_{i \leq x} 1 \dot{-} g(i, \vec{y})$$

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counts how many values $i \in \{0, \dots, x\}$ satisfy $g(i, \vec{y}) = 0$

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- ▶ $f(x, \vec{y}) = \sum_{i \leq x} 1 \dot{-} f'(i, \vec{y})$ is elementary

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elementary functions are primitive recursive

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Definition

binary function $2_x(y)$ is defined by primitive recursion

$$2_0(y) = y$$

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Example

$$2_0(3) = 3$$

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Example

$$2_0(3) = 3 \quad 2_1(3) = 2^3 = 8$$

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induction on definition of elementary functions

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► **initial functions**

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$$g(\vec{y}) < 2_c(\max\{\vec{y}\}) \quad h_i(\vec{x}) < 2_{c_i}(\max\{\vec{x}\}) \quad \text{for all } 1 \leq i \leq m$$

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$$g(\vec{y}) < 2_c(\max\{\vec{y}\}) \quad h_i(\vec{x}) < 2_{c_i}(\max\{\vec{x}\}) \quad \text{for all } 1 \leq i \leq m$$

$$\begin{aligned} f(\vec{x}) &< 2_c(\max\{h_1(\vec{x}), \dots, h_m(\vec{x})\}) < 2_c(\max\{2_{c_1}(\max\{\vec{x}\}), \dots, 2_{c_m}(\max\{\vec{x}\})\}) \\ &= 2_c(2_{\max\{c_1, \dots, c_m\}}(\max\{\vec{x}\})) \end{aligned}$$

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► **composition** $f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x}))$

$$g(\vec{y}) < 2_c(\max\{\vec{y}\}) \quad h_i(\vec{x}) < 2_{c_i}(\max\{\vec{x}\}) \quad \text{for all } 1 \leq i \leq m$$

$$\begin{aligned} f(\vec{x}) &< 2_c(\max\{h_1(\vec{x}), \dots, h_m(\vec{x})\}) < 2_c(\max\{2_{c_1}(\max\{\vec{x}\}), \dots, 2_{c_m}(\max\{\vec{x}\})\}) \\ &= 2_c(2_{\max\{c_1, \dots, c_m\}}(\max\{\vec{x}\})) = 2_{c+\max\{c_1, \dots, c_m\}}(\max\{\vec{x}\}) \end{aligned}$$

Proof (cont'd)

induction on definition of elementary functions

► **bounded summation** $f(x, \vec{y}) = \sum_{i=0}^x g(i, \vec{y})$

Proof (cont'd)

induction on definition of elementary functions

- ▶ **bounded summation** $f(x, \vec{y}) = \sum_{i=0}^x g(i, \vec{y})$ with $g(i, \vec{y}) < 2_{\text{c}}(\max\{i, \vec{y}\})$

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► **bounded summation** $f(x, \vec{y}) = \sum_{i=0}^x g(i, \vec{y})$ with $g(i, \vec{y}) < 2_c(\max\{i, \vec{y}\})$

$$f(x, \vec{y}) < (x + 1) \times 2_c(\max\{x, \vec{y}\})$$

Proof (cont'd)

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► **bounded summation** $f(x, \vec{y}) = \sum_{i=0}^x g(i, \vec{y})$ with $g(i, \vec{y}) < 2_c(\max\{i, \vec{y}\})$

$$f(x, \vec{y}) < (x + 1) \times 2_c(\max\{x, \vec{y}\}) \leq 2_2(\max\{x, \vec{y}\}) \times 2_c(\max\{x, \vec{y}\})$$

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Lemma

$$x^y \leq 2_3(\max\{x, y\})$$

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Proof

$$x^y \leq 2^{xy}$$

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► **bounded product** $f(x, \vec{y}) = \prod_{i=0}^x g(i, \vec{y})$

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$$f(x, \vec{y}) < 2_c(\max\{x, \vec{y}\})^{x+1}$$

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$$f(x, \vec{y}) < 2_c(\max\{x, \vec{y}\})^{x+1} \leq 2_c(\max\{x, \vec{y}\})^{2_1(\max\{x, \vec{y}\})}$$

Proof (cont'd)

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Proof (cont'd)

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Corollary

$E \subsetneq PR$

Proof (cont'd)

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Proof

if primitive recursive function $2_x(y)$ is elementary then $2_x(y) < 2_c(\max\{x, y\})$ for some constant c

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Proof

if primitive recursive function $2_x(y)$ is elementary then $2_x(y) < 2_c(\max\{x, y\})$ for some constant c and thus $2_c(c) < 2_c(\max\{c, c\}) = 2_c(c)$



Outline

1. Summary of Previous Lecture
2. Loop Programs
3. Elementary Functions
- 4. Grzegorzcyk Hierarchy**
5. Summary

Definition (bounded recursion)

class C of numeric functions is closed under **bounded recursion** if $f: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ defined by primitive recursion from $g: \mathbb{N}^n \rightarrow \mathbb{N} \in C$ and $h: \mathbb{N}^{n+2} \rightarrow \mathbb{N} \in C$ and satisfying

$$f(x, \vec{y}) \leq i(x, \vec{y})$$

for some $i: \mathbb{N}^{n+1} \rightarrow \mathbb{N} \in C$ different from f belongs to C

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Definitions

► $e_0(x, y) = x + y$

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Definitions

- ▶ $e_0(x, y) = x + y$ $e_1(x) = x^2 + 2$
- ▶ $e_{n+2}(x) = \begin{cases} 2 & \text{if } x = 0 \\ e_{n+1}(e_{n+2}(x-1)) & \text{if } x > 0 \end{cases}$

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Example

$$e_2(0) = 2$$

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Example

$$e_2(0) = 2 \quad e_2(1) = e_1(e_2(0)) = e_1(2) = 6$$

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$$e_2(0) = 2 \quad e_2(1) = 6 \quad e_2(2) = e_1(e_2(1))$$

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Example

$$e_2(0) = 2 \quad e_2(1) = 6 \quad e_2(2) = e_1(e_2(1)) = e_1(6) = 38$$

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Example

$$e_2(0) = 2 \quad e_2(1) = 6 \quad e_2(2) = 38 \quad e_2(3) = e_1(e_2(2)) = e_1(38) = 1446$$

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Proof

induction on x

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Definitions

- E_0 is smallest class of functions that contains all initial functions and is closed under composition and bounded recursion

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Examples

- ▶ E_0 contains $f(x) = x + 2$

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Examples

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- ▶ E_1 contains $f(x) = 4x$

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- ▶ E_0 contains $f(x) = x + 2$
- ▶ E_1 contains $f(x) = 4x$
- ▶ E_2 contains $f(x) = x^4$

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Examples

- ▶ E_0 contains $f(x) = x + 2$
- ▶ E_1 contains $f(x) = 4x$
- ▶ E_2 contains $f(x) = x^4$
- ▶ E_3 contains $f(x) = 2^{2^x}$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_0 \quad \exists c \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq \max \{x_1, \dots, x_n\} + c$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_0 \quad \exists c \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq \max \{x_1, \dots, x_n\} + c$

Proof

► $z(x) \leq x$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_0 \quad \exists c \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq \max \{x_1, \dots, x_n\} + c$

Proof

- ▶ $z(x) \leq x$
- ▶ $s(x) \leq x + 1$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_0 \quad \exists c \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq \max \{x_1, \dots, x_n\} + c$

Proof

- ▶ $z(x) \leq x$
- ▶ $s(x) \leq x + 1$
- ▶ $\pi_i^n(x_1, \dots, x_n) \leq x_i \leq \max \{x_1, \dots, x_n\}$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_0 \quad \exists c \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq \max \{x_1, \dots, x_n\} + c$

Proof

- ▶ $z(x) \leq x$
- ▶ $s(x) \leq x + 1$
- ▶ $\pi_i^n(x_1, \dots, x_n) \leq x_i \leq \max \{x_1, \dots, x_n\}$
- ▶ f is obtained by composing g and $h_1, \dots, h_m$

$$f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x}))$$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_0 \quad \exists c \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq \max \{x_1, \dots, x_n\} + c$

Proof

- ▶ $z(x) \leq x$
- ▶ $s(x) \leq x + 1$
- ▶ $\pi_i^n(x_1, \dots, x_n) \leq x_i \leq \max \{x_1, \dots, x_n\}$
- ▶ f is obtained by composing g and $h_1, \dots, h_m$

$$f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x})) \leq \max \{h_1(\vec{x}), \dots, h_m(\vec{x})\} + c_g \quad g \in E_0$$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_0 \quad \exists c \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq \max \{x_1, \dots, x_n\} + c$

Proof

- ▶ $z(x) \leq x$
- ▶ $s(x) \leq x + 1$
- ▶ $\pi_i^n(x_1, \dots, x_n) \leq x_i \leq \max \{x_1, \dots, x_n\}$
- ▶ f is obtained by composing g and $h_1, \dots, h_m$

$$\begin{aligned} f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x})) &\leq \max \{h_1(\vec{x}), \dots, h_m(\vec{x})\} + c_g && g \in E_0 \\ &\leq \max \{\vec{x}\} + \max \{c_{h_1}, \dots, c_{h_m}\} + c_g && h_1, \dots, h_m \in E_0 \end{aligned}$$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_0 \quad \exists c \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq \max \{x_1, \dots, x_n\} + c$

Proof

- ▶ $z(x) \leq x$
- ▶ $s(x) \leq x + 1$
- ▶ $\pi_i^n(x_1, \dots, x_n) \leq x_i \leq \max \{x_1, \dots, x_n\}$
- ▶ f is obtained by composing g and $h_1, \dots, h_m$

$$\begin{aligned} f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x})) &\leq \max \{h_1(\vec{x}), \dots, h_m(\vec{x})\} + c_g && g \in E_0 \\ &\leq \max \{\vec{x}\} + \max \{c_{h_1}, \dots, c_{h_m}\} + c_g && h_1, \dots, h_m \in E_0 \end{aligned}$$

- ▶ f is obtained by bounded recursion from g and h with bound i

$$f(\vec{x}) \leq i(\vec{x})$$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_0 \quad \exists c \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq \max \{x_1, \dots, x_n\} + c$

Proof

- ▶ $z(x) \leq x$
- ▶ $s(x) \leq x + 1$
- ▶ $\pi_i^n(x_1, \dots, x_n) \leq x_i \leq \max \{x_1, \dots, x_n\}$
- ▶ f is obtained by composing g and $h_1, \dots, h_m$

$$\begin{aligned} f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x})) &\leq \max \{h_1(\vec{x}), \dots, h_m(\vec{x})\} + c_g && g \in E_0 \\ &\leq \max \{\vec{x}\} + \max \{c_{h_1}, \dots, c_{h_m}\} + c_g && h_1, \dots, h_m \in E_0 \end{aligned}$$

- ▶ f is obtained by bounded recursion from g and h with bound i

$$f(\vec{x}) \leq i(\vec{x}) \leq \max \{x_1, \dots, x_n\} + c_i \quad i \in E_0$$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_1 \quad \exists$ linear function $g: \mathbb{N}^n \rightarrow \mathbb{N}$ such that $f(x_1, \dots, x_n) \leq g(x_1, \dots, x_n)$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_1 \quad \exists$ linear function $g: \mathbb{N}^n \rightarrow \mathbb{N}$ such that $f(x_1, \dots, x_n) \leq g(x_1, \dots, x_n)$

Proof

► ...

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_1 \quad \exists$ linear function $g: \mathbb{N}^n \rightarrow \mathbb{N}$ such that $f(x_1, \dots, x_n) \leq g(x_1, \dots, x_n)$

Proof

► ...

► $f = e_0 = x + y$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_1 \quad \exists$ linear function $g: \mathbb{N}^n \rightarrow \mathbb{N}$ such that $f(x_1, \dots, x_n) \leq g(x_1, \dots, x_n)$

Proof

► ...

► $f = e_0 = x + y$

► f is obtained by composing g and $h_1, \dots, h_m$

$$f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x})) \leq \ell_g(h_1(\vec{x}), \dots, h_m(\vec{x})) \quad g \in E_1$$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_1 \quad \exists$ linear function $g: \mathbb{N}^n \rightarrow \mathbb{N}$ such that $f(x_1, \dots, x_n) \leq g(x_1, \dots, x_n)$

Proof

► ...

► $f = e_0 = x + y$

► f is obtained by composing g and $h_1, \dots, h_m$

$$\begin{aligned} f(\vec{x}) &= g(h_1(\vec{x}), \dots, h_m(\vec{x})) \leq \ell_g(h_1(\vec{x}), \dots, h_m(\vec{x})) & g \in E_1 \\ &\leq \ell_g(\ell_{h_1}(\vec{x}), \dots, \ell_{h_m}(\vec{x})) & h_1, \dots, h_m \in E_1 \end{aligned}$$

Lemma

$\forall f: \mathbb{N}^n \rightarrow \mathbb{N} \in E_1 \quad \exists \text{ linear function } g: \mathbb{N}^n \rightarrow \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_n) \leq g(x_1, \dots, x_n)$

Proof

► ...

► $f = e_0 = x + y$

► f is obtained by composing g and $h_1, \dots, h_m$

$$\begin{aligned} f(\vec{x}) &= g(h_1(\vec{x}), \dots, h_m(\vec{x})) \leq \ell_g(h_1(\vec{x}), \dots, h_m(\vec{x})) & g \in E_1 \\ &\leq \ell_g(\ell_{h_1}(\vec{x}), \dots, \ell_{h_m}(\vec{x})) & h_1, \dots, h_m \in E_1 \end{aligned}$$

linear functions are closed under composition

Lemma

$$\textcircled{1} \quad e_n(x) > x$$

$$\forall n \geq 1 \quad \forall x \geq 0$$

Lemma

① $e_n(x) > x$

$$\forall n \geq 1 \forall x \geq 0$$

② $e_n(x+1) > e_n(x)$

$$\forall n \geq 1 \forall x \geq 0$$

Lemma

① $e_n(x) > x$

$$\forall n \geq 1 \forall x \geq 0$$

② $e_n(x+1) > e_n(x)$

$$\forall n \geq 1 \forall x \geq 0$$

③ $e_{n+1}(x) \geq e_n(x)$

$$\forall n \geq 1 \forall x \geq 0$$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ①

induction on $n > 0$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ①

induction on $n > 0$

► $e_1(x) = x^2 + 2$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ①

induction on $n > 0$

► $e_1(x) = x^2 + 2 > x$

Lemma

- | | |
|------------------------------------|--|
| ① $e_n(x) > x$ | $\forall n \geq 1 \forall x \geq 0$ |
| ② $e_n(x+1) > e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ③ $e_{n+1}(x) \geq e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ | $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$ |

Proof ①

induction on $n > 0$

- ▶ $e_1(x) = x^2 + 2 > x$
- ▶ induction on $x \geq 0$

Lemma

- | | |
|------------------------------------|--|
| ① $e_n(x) > x$ | $\forall n \geq 1 \forall x \geq 0$ |
| ② $e_n(x+1) > e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ③ $e_{n+1}(x) \geq e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ | $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$ |

Proof ①

induction on $n > 0$

- ▶ $e_1(x) = x^2 + 2 > x$
- ▶ induction on $x \geq 0$
 - ▶ $e_{n+1}(0) = 2$

Lemma

- | | |
|------------------------------------|--|
| ① $e_n(x) > x$ | $\forall n \geq 1 \forall x \geq 0$ |
| ② $e_n(x+1) > e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ③ $e_{n+1}(x) \geq e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ | $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$ |

Proof ①

induction on $n > 0$

- ▶ $e_1(x) = x^2 + 2 > x$
- ▶ induction on $x \geq 0$
 - ▶ $e_{n+1}(0) = 2 > 0$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ①

induction on $n > 0$

- ▶ $e_1(x) = x^2 + 2 > x$
- ▶ induction on $x \geq 0$
 - ▶ $e_{n+1}(0) = 2 > 0$
 - ▶ $e_{n+1}(x+1) = e_n(e_{n+1}(x))$

Lemma

- | | |
|------------------------------------|--|
| ① $e_n(x) > x$ | $\forall n \geq 1 \forall x \geq 0$ |
| ② $e_n(x+1) > e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ③ $e_{n+1}(x) \geq e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ | $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$ |

Proof ①

induction on $n > 0$

- ▶ $e_1(x) = x^2 + 2 > x$
- ▶ induction on $x \geq 0$
 - ▶ $e_{n+1}(0) = 2 > 0$
 - ▶ $e_{n+1}(x+1) = e_n(e_{n+1}(x)) > e_{n+1}(x)$

Lemma

- | | |
|------------------------------------|--|
| ① $e_n(x) > x$ | $\forall n \geq 1 \forall x \geq 0$ |
| ② $e_n(x+1) > e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ③ $e_{n+1}(x) \geq e_n(x)$ | $\forall n \geq 1 \forall x \geq 0$ |
| ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ | $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$ |

Proof ①

induction on $n > 0$

- ▶ $e_1(x) = x^2 + 2 > x$
- ▶ induction on $x \geq 0$
 - ▶ $e_{n+1}(0) = 2 > 0$
 - ▶ $e_{n+1}(x+1) = e_n(e_{n+1}(x)) > e_{n+1}(x) > x$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ①

induction on $n > 0$

► $e_1(x) = x^2 + 2 > x$

► induction on $x \geq 0$

► $e_{n+1}(0) = 2 > 0$

► $e_{n+1}(x+1) = e_n(e_{n+1}(x)) > e_{n+1}(x) > x \implies e_{n+1}(x+1) > x+1$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ②

induction on $n > 0$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ②

induction on $n > 0$

► $e_1(x+1) = (x+1)^2 + 2$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ②

induction on $n > 0$

► $e_1(x+1) = (x+1)^2 + 2 > x^2 + 2 = e_1(x)$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ②

induction on $n > 0$

- ▶ $e_1(x+1) = (x+1)^2 + 2 > x^2 + 2 = e_1(x)$
- ▶ $e_{n+1}(x+1) = e_n(e_{n+1}(x))$

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ②

induction on $n > 0$

- ▶ $e_1(x+1) = (x+1)^2 + 2 > x^2 + 2 = e_1(x)$
- ▶ $e_{n+1}(x+1) = e_n(e_{n+1}(x)) > e_{n+1}(x)$ by ①

Lemma

- ① $e_n(x) > x$ $\forall n \geq 1 \forall x \geq 0$
- ② $e_n(x+1) > e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ③ $e_{n+1}(x) \geq e_n(x)$ $\forall n \geq 1 \forall x \geq 0$
- ④ $e_{n+1}(x+k) \geq e_n^{(k)}(x)$ $\forall n \geq 1 \forall x \geq 0 \forall k \geq 0$

Proof ②

induction on $n > 0$

- ▶ $e_1(x+1) = (x+1)^2 + 2 > x^2 + 2 = e_1(x)$
- ▶ $e_{n+1}(x+1) = e_n(e_{n+1}(x)) > e_{n+1}(x)$ by ①

Proof ③ ④

homework exercise

Lemma

$\forall f: \mathbb{N}^m \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_m) \leq e_{n+1}^{(k)}(\max\{x_1, \dots, x_m\})$

Proof

► $z(x) = 0$

Lemma

$\forall f: \mathbb{N}^m \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_m) \leq e_{n+1}^{(k)}(\max\{x_1, \dots, x_m\})$

Proof

► $z(x) = 0 \leq x$

Lemma

$\forall f: \mathbb{N}^m \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_m) \leq e_{n+1}^{(k)}(\max\{x_1, \dots, x_m\})$

Proof

► $z(x) = 0 \leq x = e_{n+1}^{(0)}(x)$

Lemma

$\forall f: \mathbb{N}^m \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_m) \leq e_{n+1}^{(k)}(\max\{x_1, \dots, x_m\})$

Proof

- ▶ $z(x) = 0 \leq x = e_{n+1}^{(0)}(x)$
- ▶ $s(x) = x + 1$

Lemma

$\forall f: \mathbb{N}^m \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_m) \leq e_{n+1}^{(k)}(\max\{x_1, \dots, x_m\})$

Proof

- ▶ $z(x) = 0 \leq x = e_{n+1}^{(0)}(x)$
- ▶ $s(x) = x + 1 \leq x^2 + 2 = e_1(x)$

Lemma

$\forall f: \mathbb{N}^m \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_m) \leq e_{n+1}^{(k)}(\max\{x_1, \dots, x_m\})$

Proof

- ▶ $z(x) = 0 \leq x = e_{n+1}^{(0)}(x)$
- ▶ $s(x) = x + 1 \leq x^2 + 2 = e_1(x) \leq e_{n+1}^{(1)}(x)$

Lemma

$\forall f: \mathbb{N}^m \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_m) \leq e_{n+1}^{(k)}(\max\{x_1, \dots, x_m\})$

Proof

- ▶ $z(x) = 0 \leq x = e_{n+1}^{(0)}(x)$
- ▶ $s(x) = x + 1 \leq x^2 + 2 = e_1(x) \leq e_{n+1}^{(1)}(x)$
- ▶ $\pi_i^n(x_1, \dots, x_n) = x_i$

Lemma

$\forall f: \mathbb{N}^m \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_m) \leq e_{n+1}^{(k)}(\max\{x_1, \dots, x_m\})$

Proof

- ▶ $z(x) = 0 \leq x = e_{n+1}^{(0)}(x)$
- ▶ $s(x) = x + 1 \leq x^2 + 2 = e_1(x) \leq e_{n+1}^{(1)}(x)$
- ▶ $\pi_i^n(x_1, \dots, x_n) = x_i < e_1(x_i)$

Lemma

$\forall f: \mathbb{N}^m \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x_1, \dots, x_m) \leq e_{n+1}^{(k)}(\max\{x_1, \dots, x_m\})$

Proof

- ▶ $z(x) = 0 \leq x = e_{n+1}^{(0)}(x)$
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Proof (cont'd)

► f is obtained by composing g and $h_1, \dots, h_j$

$$f(\vec{x}) = g(h_1(\vec{x}), \dots, h_j(\vec{x}))$$

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Corollary

$\forall f: \mathbb{N} \rightarrow \mathbb{N} \in \mathbf{E}_{n+2} \quad \exists k \in \mathbb{N} \quad \text{such that} \quad f(x) \leq e_{n+2}(k + x)$

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Proof

$$f(x) \leq e_{n+1}^{(k)}(x) \leq e_{n+2}(k + x)$$

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$$e_{n+2}(x + x) \leq e_{n+2}(k + x)$$

for some $k \in \mathbb{N}$

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① $E_0 \subseteq E_1 \subseteq E_2$ by definition

$E_{n+2} \subseteq E_{n+3}$ for $n \geq 0$ because $e_{n+1} \in E_{n+3}$ by bounded recursion:

$$e_{n+1}(x) \leq e_{n+2}(x) \in E_{n+3}$$

Outline

1. Summary of Previous Lecture
2. Loop Programs
3. Elementary Functions
4. Grzegorzcyk Hierarchy
- 5. Summary**

Important Concepts

- ▶ $2_x(y)$
- ▶ bounded recursion
- ▶ $e_0, e_1, e_2, \dots$
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homework for November 9