



Computability Theory

Aart Middeldorp

Outline

- 1. Summary of Previous Lecture**
- 2. Gödel Numbering**
- 3. Course-of-Values Recursion**
- 4. Ackermann Function**
- 5. Diagonalization**
- 6. Total Recursive Functions**
- 7. Summary**

Definition

class **PR** of **primitive recursive functions** is smallest class of total functions $f: \mathbb{N}^n \rightarrow \mathbb{N}$ that contains all **initial functions**

- ▶ **zero** $z(x) = 0$
- ▶ **successor** $s(x) = x + 1$
- ▶ **projection** $\pi_i^n(x_1, \dots, x_n) = x_i$ for all $n \geq 1$ and $1 \leq i \leq n$

and is closed under **composition**

- ▶ $f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x})) \in \text{PR}$ for all $g: \mathbb{N}^m \rightarrow \mathbb{N} \in \text{PR}$ and $h_1, \dots, h_m: \mathbb{N}^n \rightarrow \mathbb{N} \in \text{PR}$

and **primitive recursion**

- ▶ $f(x, \vec{y}): \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ defined by

$$f(0, \vec{y}) = g(\vec{y})$$

$$f(x+1, \vec{y}) = h(f(x, \vec{y}), x, \vec{y})$$

belongs to PR for all $g: \mathbb{N}^n \rightarrow \mathbb{N} \in \text{PR}$ and $h: \mathbb{N}^{n+2} \rightarrow \mathbb{N} \in \text{PR}$

Definition

predicate $P: \mathbb{N}^n \rightarrow \mathbb{B}$ is primitive recursive if its **characteristic function** $\chi_P: \mathbb{N}^n \rightarrow \mathbb{N}$

$$\chi_P(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } P(x_1, \dots, x_n) \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive

Lemma

PR is closed under **iteration**, **case analysis** and **bounded minimization**

Definitions

► **pairing** function $\pi(x, y) = 2^x(2y + 1) \dot{-} 1$

► **extraction** functions

$$\pi_1(z) = (\mu x \leq z) (\exists y \leq z) [z = \pi(x, y)] \quad \pi_2(z) = (\mu y \leq z) (\exists x \leq z) [z = \pi(x, y)]$$

Part I: Recursive Function Theory

Ackermann function, bounded minimization, bounded recursion, course-of-values recursion, diagonalization, diophantine sets, elementary functions, fixed point theorem, Fibonacci numbers, Gödel numbering, Gödel's β function, Grzegorczyk hierarchy, loop programs, minimization, normal form theorem, partial recursive functions, primitive recursion, recursive enumerability, recursive inseparability, s-m-n theorem, total recursive functions, undecidability, while programs

Part II: Combinatory Logic

abstraction, arithmetization, CL-representability, combinators, combinatorial completeness, Church numerals, Church-Rosser theorem, Curry-Howard isomorphism, fixed point theorem, intuitionistic propositional logic, normalization theorem, termination, typing, undecidability, Z property

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Definitions

- **Gödel number** of sequence $x_1, \dots, x_n$: $\langle x_1, \dots, x_n \rangle = p_0^n \times p_1^{x_1} \times \dots \times p_n^{x_n}$

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- ▶ $(x)_i = (\mu j < x) \neg (p_i^{j+1} \mid x)$

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- ▶ $\text{len}(x) = (x)_0$

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- ▶ $\text{seq}(x) \iff x > 0 \wedge (\forall i \leq x) [(x)_i \neq 0 \implies i \leq \text{len}(x)]$

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- ▶ $x ; y = p_0^{\text{len}(x) + \text{len}(y)} \times \prod_{i < \text{len}(x)} p_{i+1}^{(x)_{i+1}} \times \prod_{i < \text{len}(y)} p_{\text{len}(x) + i + 1}^{(y)_{i+1}}$

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Lemma

- ▶ $x = \langle x_1, \dots, x_n \rangle \implies \text{len}(x) = n \wedge (x)_i = x_i \text{ for all } 1 \leq i \leq n$

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- ▶ $x = \langle x_1, \dots, x_n \rangle \wedge y = \langle y_1, \dots, y_m \rangle \implies x ; y = \langle x_1, \dots, x_n, y_1, \dots, y_m \rangle$

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if $f: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ then $\tilde{f}(x, \vec{y}) = \langle f(0, \vec{y}), \dots, f(x, \vec{y}) \rangle$

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Lemma

if $g: \mathbb{N}^n \rightarrow \mathbb{N}$ and $h: \mathbb{N}^{n+2} \rightarrow \mathbb{N}$ are primitive recursive then so is $f: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ defined as

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Proof

$$f(x, \vec{y}) = (\tilde{f}(x, \vec{y}))_{x+1}$$

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Proof

$f(x, \vec{y}) = (\tilde{f}(x, \vec{y}))_{x+1}$ and $\tilde{f}$ is primitive recursive:

$$\tilde{f}(0, \vec{y}) = \langle f(0, \vec{y}) \rangle = \langle g(\vec{y}) \rangle$$

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with $h'(x, y, \vec{z}) = x ; \langle h(x, y, \vec{z}) \rangle$

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$$\text{ack}(0, y) = y + 1$$

$$\text{ack}(x + 1, 0) = \text{ack}(x, 1)$$

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Example

► $\text{ack}(1, 3)$

Definition

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Example

► $\text{ack}(1, 3) = \text{ack}(0, \text{ack}(1, 2))$

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Example

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Example

► $\text{ack}(1, 3) = \text{ack}(0, 4)$

Definition

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Example

► $\text{ack}(1, 3) = 5$

Definition

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Example

- ▶ $\text{ack}(1, 3) = 5$
- ▶ $\text{ack}(2, 2)$

Definition

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Example

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Example

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Example

- ▶ $\text{ack}(1, 3) = 5$
- ▶ $\text{ack}(2, 2) = \text{ack}(0, \text{ack}(1, 4))$

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Example

- ▶ $\text{ack}(1, 3) = 5$
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Example

- ▶ $\text{ack}(1, 3) = 5$
- ▶ $\text{ack}(2, 2) = \text{ack}(0, 6)$

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Example

- ▶ $\text{ack}(1, 3) = 5$
- ▶ $\text{ack}(2, 2) = 7$

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Example

- ▶ $\text{ack}(1, 3) = 5$
- ▶ $\text{ack}(2, 2) = 7$
- ▶ $\text{ack}(3, 4) = 125$

Definition

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Example

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- ▶ $\text{ack}(2, 2) = 7$
- ▶ $\text{ack}(3, 4) = 125$
- ▶ $\text{ack}(4, 1) =$

Definition

$$\text{ack}(0, y) = y + 1$$

$$\text{ack}(x + 1, 0) = \text{ack}(x, 1)$$

$$\text{ack}(x + 1, y + 1) = \text{ack}(x, \text{ack}(x + 1, y))$$



Example

- ▶ $\text{ack}(1, 3) = 5$
- ▶ $\text{ack}(2, 2) = 7$
- ▶ $\text{ack}(3, 4) = 125$
- ▶ $\text{ack}(4, 1) = 65533$

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Lemma

Ackermann function is well-defined

Example (cont'd)

► $\text{ack}(4, 3) = 2^{2^{65533}} - 3$

Example (cont'd)

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Remarks

- Ackermann function grows very fast

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Remarks

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- inverse of Ackermann function appears in complexity analysis of certain algorithms

Example (cont'd)

► $\text{ack}(4, 3) = 2^{2^{65533}} - 3$

Remarks

- Ackermann function grows very fast
- Ackermann function is used as compiler benchmark
- inverse of Ackermann function appears in complexity analysis of certain algorithms

Lemma

- 1 $\text{ack}(x, y) > y$
- 2 $\text{ack}(x, y + 1) > \text{ack}(x, y)$
- 3 $\text{ack}(x + 1, y) \geq \text{ack}(x, y + 1)$
- 4 $\text{ack}(x + 2, y) > \text{ack}(x, 2y)$

► skip proofs

Lemma ①

► $\text{ack}(x, y) > y$

Proof

induction on x

► $\text{ack}(0, y) = y + 1 > y$

Lemma ①

- ▶ $\text{ack}(x, y) > y$

Proof

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- ▶ $\text{ack}(0, y) = y + 1 > y$
- ▶ induction on y

$$\text{ack}(x + 1, 0) = \text{ack}(x, 1)$$

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- ▶ $\text{ack}(0, y) = y + 1 > y$
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► $\text{ack}(x, y) > y$

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► $\text{ack}(0, y) = y + 1 > y$

► induction on y

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Lemma 2

► $\text{ack}(x, y + 1) > \text{ack}(x, y)$

Proof

induction on x

$$\text{ack}(0, y + 1) = y + 2 > y + 1 = \text{ack}(0, y)$$

Lemma 2

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Lemma 3

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Proof

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► $\text{ack}(x + 1, y) \geq \text{ack}(x, y + 1)$

Proof

induction on x

► $\text{ack}(1, y) \geq y + 2 = \text{ack}(0, y + 1)$ by induction on y

Lemma 3

$$\triangleright \text{ack}(x+1, y) \geq \text{ack}(x, y+1)$$

Proof

induction on x

$$\triangleright \text{ack}(1, y) \geq y+2 = \text{ack}(0, y+1) \text{ by induction on } y$$

$$\text{ack}(1, 0) = \text{ack}(0, 1) = 2$$

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$\triangleright$ induction on y

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Lemma 3

► $\text{ack}(x + 1, y) \geq \text{ack}(x, y + 1) > \text{ack}(x, y)$

Corollary

Ackermann function is strictly monotone in both arguments

Lemma 4

► $\text{ack}(x + 2, y) > \text{ack}(x, 2y)$

Proof

induction on x

Lemma 4

► $\text{ack}(x + 2, y) > \text{ack}(x, 2y)$

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► $\text{ack}(2, y) > \text{ack}(0, 2y)$ by induction on y

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$$\text{ack}(2, y) > \text{ack}(0, 2y) > 2y$$

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$$\text{ack}(2, y) > 2y + 1$$

Lemma 4

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Lemma 4

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$$\geq \text{ack}(x + 2, \text{ack}(x, 2y + 1)) > \text{ack}(x + 2, 2y + 1)$$

$$\geq \text{ack}(x + 1, 2y + 2)$$

Lemma

$\forall$ primitive recursive function $f: \mathbb{N}^n \rightarrow \mathbb{N}$ $\exists$ constant $c \in \mathbb{N}$ such that

$$f(x_1, \dots, x_n) < \text{ack}(c, \max \{x_1, \dots, x_n\})$$

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Proof

induction on definition of primitive recursive functions

Lemma

$\forall$ primitive recursive function $f: \mathbb{N}^n \rightarrow \mathbb{N} \quad \exists$ constant $c \in \mathbb{N}$ such that

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Proof

induction on definition of primitive recursive functions

► initial functions

$$z(x) = 0 < x + 1 = \text{ack}(0, x)$$

Lemma

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Ackermann function is **not** primitive recursive

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if Ackermann function is primitive recursive then

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Outline

1. Summary of Previous Lecture
2. Gödel Numbering
3. Course-of-Values Recursion
4. Ackermann Function
- 5. Diagonalization**
6. Total Recursive Functions
7. Summary

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Examples

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Remark (key insight)

from index we can reconstruct function

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- ▶ $\ulcorner f \urcorner = \langle 4, \ulcorner g \urcorner, \ulcorner h \urcorner \rangle$ if f is obtained by primitive recursion from g and h

Examples

- ▶ $\ulcorner + \urcorner = \langle 4, \langle 2, 1, 1 \rangle, \langle 3, \langle 1 \rangle, \langle 2, 3, 1 \rangle \rangle \rangle$
- ▶ $397065375000 = 2^3 \times 3^3 \times 5^6 \times 7^6 = \langle 3, \langle 1 \rangle, \langle 1 \rangle \rangle = \ulcorner s \circ s \urcorner$

Remark (key insight)

from index we can reconstruct function

(details in later lecture)

Corollary

$\exists$ computable list $f_0, f_1, f_2, f_3, \dots$ of unary primitive recursive functions

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Diagonalization

$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	$\dots$
$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$\dots$
$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$\dots$
$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Corollary

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Diagonalization

$f_0(0)$ $f_0(1)$ $f_0(2)$ $f_0(3)$ $\dots$

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$$g(x) = f_x(x) + 1$$

Corollary

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$$g(x) = f_x(x) + 1$$

Corollary

function g is computable but not primitive recursive

Outline

1. Summary of Previous Lecture
2. Gödel Numbering
3. Course-of-Values Recursion
4. Ackermann Function
5. Diagonalization
- 6. Total Recursive Functions**
7. Summary

Definition

class **R** of **recursive functions** is smallest class of total functions that contains all initial functions and is closed under composition, primitive recursion, and **(unbounded) minimization**:

$$(\mu i) P(i, \vec{y}) = \min \{i \mid P(i, \vec{y})\} \in R$$

for all $P: \mathbb{N}^{n+1} \rightarrow \mathbb{B}$ such that $\chi_P \in R$

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Theorem

Ackermann function is recursive

$$\text{ack}(0, y) = y + 1$$

$$\text{ack}(x + 1, 0) = \text{ack}(x, 1)$$

$$\text{ack}(x + 1, y + 1) = \text{ack}(x, \text{ack}(x + 1, y))$$

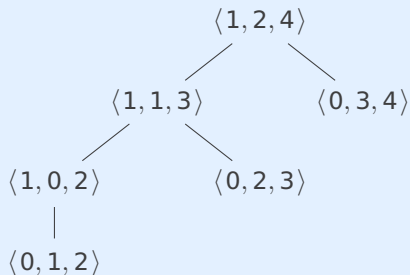
$$\text{ack}(1, 2) = 4$$

$$\text{ack}(0, y) = y + 1$$

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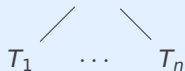
$\text{ack}(1, 2) = 4$:



$\langle x, y, z \rangle$ denotes $\text{ack}(x, y) = z$

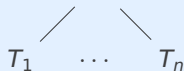
Definition

encode computation tree $T = \langle x, y, z \rangle$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$



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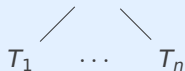
Lemma

$\exists$ primitive recursive predicate $P: \mathbb{N} \rightarrow \mathbb{B}$ such that

$$P(x) \iff x \text{ encodes correct computation tree}$$

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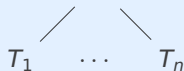
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Corollary

► $\text{ack}(x, y) = z \iff \exists t \text{ such that } P(t) \text{ and } (t)_1 = \langle x, y, z \rangle$

Definition

encode computation tree $T = \langle x, y, z \rangle$ as number $\ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$



Lemma

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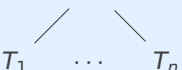
$$P(x) \iff x \text{ encodes correct computation tree}$$

Corollary

- $\text{ack}(x, y) = z \iff \exists t \text{ such that } P(t) \text{ and } (t)_1 = \langle x, y, z \rangle$
- $\text{ack}(x, y) = (\mu t) (P(t) \wedge (t)_{1,1} = x \wedge (t)_{1,2} = y) 1, 3$

Proof

$$\triangleright P(x) \iff \text{seq}(x) \wedge \text{seq}((x)_1) \wedge \text{len}((x)_1) = 3 \wedge (A(x) \vee B(x) \vee C(x))$$

$$T = \langle x, y, z \rangle \quad \ulcorner T \urcorner = \langle \langle x, y, z \rangle, \ulcorner T_1 \urcorner, \dots, \ulcorner T_n \urcorner \rangle$$


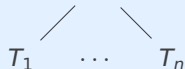
The diagram shows a tree structure. At the top is the expression $T = \langle x, y, z \rangle$. Below it, two diagonal lines branch out to T_1 on the left and T_n on the right. Between T_1 and T_n is an ellipsis $\dots$.

Proof

- ▶ $P(x) \iff \text{seq}(x) \wedge \text{seq}((x)_1) \wedge \text{len}((x)_1) = 3 \wedge (A(x) \vee B(x) \vee C(x))$
- ▶ $A(x) \iff \text{len}(x) = 1 \wedge (x)_{1,1} = 0 \wedge (x)_{1,3} = s((x)_{1,2})$

$$\text{ack}(0, y) = y + 1$$

$$T = \langle x, y, z \rangle \quad \ulcorner T \urcorner = \langle \langle 0, y, y + 1 \rangle \rangle$$

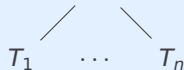


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- ▶ $B(x) \iff \text{len}(x) = 2 \wedge (x)_{1,1} > 0 \wedge (x)_{1,2} = 0 \wedge (x)_{2,1,1} = p((x)_{1,1}) \wedge (x)_{2,1,2} = 1 \wedge (x)_{2,1,3} = (x)_{1,3} \wedge P((x)_2)$

$$\text{ack}(x+1, 0) = \text{ack}(x, 1)$$

$$T = \langle x, y, z \rangle \quad \ulcorner T \urcorner = \langle \langle x+1, 0, z \rangle, \langle \langle x, 1, z \rangle, \dots \rangle \rangle$$

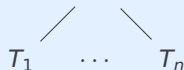


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- ▶ $C(x) \iff \text{len}(x) = 3 \wedge (x)_{1,1} > 0 \wedge (x)_{1,2} > 0 \wedge (x)_{2,1,1} = (x)_{1,1} \wedge (x)_{2,1,2} = p((x)_{1,2}) \wedge (x)_{3,1,1} = p((x)_{1,1}) \wedge (x)_{3,1,2} = (x)_{2,1,3} \wedge (x)_{3,1,3} = (x)_{1,3} \wedge P((x)_2) \wedge P((x)_3)$

$$\text{ack}(x+1, y+1) = \text{ack}(x, \text{ack}(x+1, y))$$

$$T = \langle x, y, z \rangle \quad \ulcorner T \urcorner = \langle \langle x+1, y+1, z \rangle, \langle \langle x+1, y, w \rangle, \dots \rangle, \langle \langle x, w, z \rangle, \dots \rangle \rangle$$



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Important Concepts

- ▶ Ackermann function
- ▶ course-of-values recursion
- ▶ diagonalization
- ▶ Gödel numbering
- ▶ index
- ▶ R
- ▶ recursive function
- ▶ (unbounded) minimization

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homework for October 19