universität innsbruck

Computability Theory	26W	LVA 703317
Lecture 5		November 23, 2026

All Solutions

1 The idea is to construct a computation tree where the children of the root note $\langle\langle 5, \ulcorner g \urcorner\rangle, \langle \vec{y} \rangle, n\rangle$ are labeled with

$$\langle \ulcorner g \urcorner, \langle 0, \vec{y}\rangle, z_0\rangle \qquad \langle \ulcorner g \urcorner, \langle 1, \vec{y}\rangle, z_1\rangle \qquad \ldots \qquad \langle \ulcorner g \urcorner, \langle n, \vec{y}\rangle, z_n\rangle$$

such that $z_n = 0$ and $z_0, \ldots, z_{n-1} > 0$. The local dependencies between the nodes give rise to the following primitive recursive predicate $\mathsf{minimization}(x)$:

$$\begin{aligned}&\mathsf{seq}(x) \wedge \mathsf{triple}((x)_1) \wedge \mathsf{seq}((x)_{1,1}) \wedge \mathsf{len}((x)_{1,1}) = 2 \wedge (x)_{1,1,1} = 5 \wedge \mathsf{seq}((x)_{1,2})\\ &\wedge \mathsf{len}(x) = (x)_{1,3} + 2 \wedge (\forall\, i \leqslant (x)_{1,3}) \left[(x)_{i+2,1,1} = (x)_{1,1,2} \wedge (x)_{i+2,1,2} = \langle i \rangle ; (x)_{1,2} \right]\\ &\wedge (\forall\, i < (x)_{1,3}) \left[(x)_{i+2,1,3} > 0 \right] \wedge (x)_{\mathsf{len}(x),1,3} = 0\end{aligned}$$

2 We define the function *arity* and a predicate *index* that tests whether its argument is the index of a partial recursive function by simultaneous primitive recursion:

$$\mathsf{arity}(x) = \begin{cases} 0 & \text{if } \neg\, \mathsf{index}(x)\\ 1 & \text{if } (x)_1 = 0 \text{ or } (x)_1 = 1\\ (x)_2 & \text{if } (x)_1 = 2\\ \mathsf{arity}((x)_3) & \text{if } (x)_1 = 3\\ \mathsf{arity}((x)_2) + 1 & \text{if } (x)_1 = 4\\ \mathsf{arity}((x)_2) \mathbin{\dot{-}} 1 & \text{if } (x)_1 = 5 \end{cases}$$

with

$$\begin{aligned}\mathsf{index}(x) \iff\ & x = \langle 0 \rangle \vee x = \langle 1 \rangle \vee (\exists\, y \leqslant x)\, (\exists\, z \leqslant x) \big(\\ & \left[x = \langle 2, y, z \rangle \wedge y > 0 \wedge z \geqslant 1 \wedge y \geqslant z \right]\\ \vee\ & \big[x = \langle 3, y \rangle ; z \wedge \mathsf{index}(y) \wedge \mathsf{seq}(z) \wedge \mathsf{len}(z) = \mathsf{arity}(y) \wedge\\ & \quad (\forall\, i < \mathsf{len}(z))\, [\, \mathsf{index}((z)_{i+1}) \wedge \mathsf{arity}((z)_{i+1}) = \mathsf{arity}((z)_1)\,] \big]\\ \vee\ & \left[x = \langle 4, y, z \rangle \wedge \mathsf{index}(y) \wedge \mathsf{index}(z) \wedge \mathsf{arity}(z) = \mathsf{arity}(y) + 2 \right]\\ \vee\ & \left[x = \langle 5, y \rangle \wedge \mathsf{index}(y) \wedge \mathsf{arity}(y) \geqslant 1 \right] \big)\end{aligned}$$

3 (a) Consider the partial recursive function

$$\varphi(x, y) = (\mu\, i)\, (1 \mathbin{\dot{-}} (x \mathbin{\dot{-}} i) = 0)$$

We have $\varphi(0, y)\!\uparrow$ and $\varphi(x, y) = 0$ for all $x \geqslant 1$ and $y \geqslant 0$. Hence $(\mu\, i)\, (\varphi(i, y) = 0)$ is everywhere undefined whereas $(\nu\, i)\, (\varphi(i, y) = 0)$ is the constant one function.

(b) Define the functions ψ_1 and ψ_2 as follows:

$$\psi_1(i,x) \;=\; \begin{cases} \varphi_x(x)\cdot 0 & \text{if } i=0 \\ 0 & \text{if } i>0 \end{cases} \qquad\qquad \psi_2(x) \;=\; 1 \mathbin{\dot{-}} (\nu\, i)\,(\psi_1(i,x)=0)$$

The function ψ_1 is partial recursive. Suppose for a proof by contradiction that the class of partial recursive functions is closed under the ν operation. Then ψ_2 is partial recursive. Since $\psi_1(1,x)=0$, ψ_2 is a total function and hence recursive. We have

$$\psi_2(x) \;=\; 1 \quad\Longleftrightarrow\quad (\nu\, i)\,(\psi_1(i,x)=0) \;=\; 0 \quad\Longleftrightarrow\quad \varphi_x(x)\!\downarrow$$

Since the latter is undecidable, ψ_2 cannot be recursive.

4 Assume to the contrary that *recursive* is primitive recursive. Define the predicate *defined* as

$$\mathsf{defined}(f,x) \quad\Longleftrightarrow\quad \mathsf{recursive}(\langle 3, f, c(x)\rangle)$$

where $c(x)$ is the index of the (unary) constant x function $(\mathsf{s}^{(x)} \circ \mathsf{z})$:

$$\begin{aligned} c(0) &= \langle 0\rangle \\ c(x+1) &= \langle 3, \langle 1\rangle, c(x)\rangle \end{aligned}$$

Consider the partial recursive function f defined as

$$f(x) \;=\; (\mu\, i)\,(\chi_{\mathsf{defined}}(x,x)=0)$$

and let e be its index. We have

$$\mathsf{defined}(e,e) \quad\Longleftrightarrow\quad \mathsf{recursive}(\langle 3, e, c(e)\rangle) \quad\Longleftrightarrow\quad \varphi_e(e)\!\downarrow \quad\Longleftrightarrow\quad \chi_{\mathsf{defined}}(e,e)=0$$

which is impossible.

5 First we prove the existence of a unary (primitive) recursive function h such that $\varphi_{f(h(x),x)}(y) \simeq \varphi_{h(x)}(y)$. Consider the partial recursive function

$$\psi(x,y,z) \;=\; \begin{cases} \varphi_{\varphi^2_x(x,y)}(z) & \text{if } \varphi^2_x(x,y)\!\downarrow \\ \text{undefined} & \text{otherwise} \end{cases}$$

According to the s-m-n theorem $\psi(x,y,z) \simeq \varphi_{k(x,y)}(z)$ for some primitive recursive function k. Let e be an index of the recursive function $f(k(x,y),y)$. So $\varphi^2_e(x,y) = f(k(x,y),y)$. We define $h(x) = k(e,x)$. We have

$$\varphi_{h(x)}(y) \;=\; \varphi_{k(e,x)}(y) \;\simeq\; \psi(e,x,y) \;=\; \varphi_{\varphi^2_e(e,x)}(y) \;=\; \varphi_{f(k(e,x),x)}(y) \;=\; \varphi_{f(h(x),x)}(y)$$

as desired. Consider now the recursive function $g(h(x),x)$. According to the fixed point theorem there exists a number $b \in \mathbb{N}$ such that $\varphi_b \simeq \varphi_{g(h(b),b)}$. Let $a = h(b)$. We have $\varphi_{f(a,b)} = \varphi_{f(h(b),b)} \simeq \varphi_{h(b)} = \varphi_a$ and $\varphi_{g(a,b)} = \varphi_{g(h(b),b)} \simeq \varphi_b$.