universität innsbruck

Computability Theory	26W	LVA 703317
Week 4		November 16, 2026

All Solutions

1 The following while program $P(x;y)$ computes $\mathsf{log}(x) = (\mu\, y)\,(x = 2^y)$:

$y := 0;$
$z := 0;\ z{+}{+};$ $\quad (\, z \leftarrow 2^y \,)$
$o := 0;$
$P_{=}(x, z; w);$ $\quad (\, w \leftarrow \chi_{=}(x, z) \,)$
$P_{=}(w, o; v);$ $\quad (\, v \leftarrow \chi_{\neq}(x, z) \,)$
WHILE $v > 0$ DO
  $y{+}{+};$
  $P_{+}(z, z; z);$
  $P_{=}(x, z; w);$
  $P_{=}(w, o; v)$
OD

2 We have $f(x) = (\mu\, y)\,(y = 2 \wedge \neg\,(2 \mid x))$.

3 The following while program $P(x;y)$ computes the given function:

$y := 0;$ $\quad (\, y \leftarrow 0 \,)$
$P_{\times}(y, y; u);$ $\quad (\, u \leftarrow y^2 \,)$
$P_{\dot{-}}(x, u; v);$ $\quad (\, v \leftarrow x \mathbin{\dot{-}} y^2 \,)$
WHILE $v > 0$ DO
  $y{+}{+};$
  $P_{\times}(y, y; u);$
  $P_{\dot{-}}(x, u; v);$
OD; $\quad (\, y^2 \geqslant x \,)$
$P_{\dot{-}}(u, x; v);$ $\quad (\, v \leftarrow y^2 \mathbin{\dot{-}} x \,)$
$z := 0;$
$z{+}{+};$ $\quad (\, z \leftarrow 1 \,)$
$P_{\dot{-}}(z, v; u);$ $\quad (\, u \leftarrow 1 \mathbin{\dot{-}} (y^2 \mathbin{\dot{-}} x) \,)$
WHILE $u > 0$ DO $\quad (\, u > 0 \implies y^2 = x \,)$
  $u{+}{+};$
OD;
$P_{\mathsf{p}}(x; y)$ $\quad (\, y \leftarrow x \mathbin{\dot{-}} 1 \,)$

Here $P_{\times}$ and $P_{\dot{-}}$ are programs that compute multiplication and the monus function, and P_{p} is a program that computes the predecessor function.

4 Multiplication can be defined by primitive recursion. We have $xy \leqslant (x+y)^2 + 2 = \mathsf{e}_1(\mathsf{e}_0(x,y))$ and hence $xy \in \mathsf{E}_2$. Moreover, $xy \notin \mathsf{E}_1$ since there are no constants a, b and c such that $xy \leqslant ax + by + c$.

5 Suppose f is defined from g and h in E by bounded recursion, so

$$\begin{aligned} f(0,\vec{y}) &= g(\vec{y}) \\ f(x+1,\vec{y}) &= h(f(x,\vec{y}),x,\vec{y}) \\ f(x,\vec{y}) &\leqslant k(x,\vec{y}) \end{aligned}$$

with $k \in \mathsf{E}$. The idea is to encode the sequence of calls $f(0,\vec{y}),\ldots,f(x,\vec{y})$ as

$$\tilde{f}(x,\vec{y}) = \mathsf{p}_0^{f(0,\vec{y})} \times \cdots \times \mathsf{p}_x^{f(x,\vec{y})}$$

from which $f(x,\vec{y})$ can be extracted by $\mathsf{exp}(\tilde{f}(x,\vec{y}),\mathsf{p}_x)$. An (elementary) upper bound on $\tilde{f}(x,\vec{y})$ is easy to give:

$$\tilde{f}(x,\vec{y}) \leqslant (p_x^{l(x,\vec{y})})^{x+1} \quad \text{with} \quad l(x,\vec{y}) = \sum_{i \leqslant x} k(i,\vec{y})$$

We have

$$f(x,\vec{y}) = (\mu\, z \leqslant k(x,\vec{y}))\, P_f(x,\vec{y},z)$$

with

$$P_f(x,\vec{y},z) \iff \left(\exists j \leqslant (p_x^{l(x,\vec{y})})^{x+1}\right) \left[\begin{array}{l} \mathsf{exp}(j,\mathsf{p}_0) = g(\vec{y}) \wedge z = \mathsf{exp}(j,\mathsf{p}_x) \wedge \\ (\forall\, i < x)\,(\mathsf{exp}(j,\mathsf{p}_{i+1}) = h(\mathsf{exp}(j,\mathsf{p}_i),i,\vec{y}) \end{array}\right]$$

All operations involved are elementary (preserving) and hence it follows that $f \in \mathsf{E}$.

6 First we prove $\mathsf{E} \subseteq \mathsf{E}_3$. Since $\dot{-}$ can be defined by bounded recursion

$$\begin{aligned} x \dot{-} 0 &= x & \mathsf{p}(0) &= 0 \\ x \dot{-} (y+1) &= \mathsf{p}(x \dot{-} y) & \mathsf{p}(x+1) &= x \\ x \dot{-} y &\leqslant x & \mathsf{p}(x) &\leqslant x \end{aligned}$$

it remains to show that E_3 is closed under bounded summation and bounded product. Both can be defined by primitive recursion, so we only need to show that they are bounded by functions in E_3. Since

$$\sum_{i \leqslant x} f(i,\vec{y}) \leqslant (x+1) \times \mathsf{max}\{\, f(i,\vec{y}) \mid i \leqslant x \,\}$$

and

$$\prod_{i \leqslant x} f(i,\vec{y}) \leqslant \mathsf{max}\{\, f(i,\vec{y}) \mid i \leqslant x \,\}^{x+1}$$

and E_3 contains multiplication ($x \times y \leqslant \mathsf{e}_1(\mathsf{e}_0(x,y)) \leqslant \mathsf{e}_2(\mathsf{e}_0(x,y))$) and exponentiation (see below), we must show that E_3 is closed under bounded maximum. We have

$$\mathsf{max}\{\, f(i,\vec{y}) \mid i \leqslant x \,\} = f(h(x,\vec{y}),\vec{y})$$

with

$$\begin{aligned} h(0,\vec{y}) &= 0 \\ h(x+1,\vec{y}) &= \begin{cases} h(x,\vec{y}) & \text{if } f(x+1,\vec{y}) \leqslant f(h(x,\vec{y}),\vec{y}) \\ x+1 & \text{otherwise} \end{cases} \\ &= \mathsf{ITE}(f(x+1,\vec{y}) \dot{-} f(h(x,\vec{y}),\vec{y}), x+1, h(x,\vec{y})) \\ h(x,\vec{y}) &\leqslant x \end{aligned}$$

and where ITE is defined by bounded recursion:

$$\begin{aligned} \mathsf{ITE}(0, y, z) &= y \\ \mathsf{ITE}(x+1, y, z) &= z \\ \mathsf{ITE}(x, y, z) &\leqslant y + z \end{aligned}$$

It remains to show that exponentation belongs to E_3. Since exponentiation is primitive recursive, it suffices to give an upper bound in E_3. First we prove by induction on x that $2_2(x) \leqslant \mathsf{e}_2(x)$. For $x = 0$ we have $2 = 2$. For the induction step we have $2_2(x+1) < 2_2(x)^2 + 2 \leqslant \mathsf{e}_2(x)^2 + 2 \leqslant \mathsf{e}_2(x+1)$. Hence

$$x^y \leqslant 2^{x \times y} \leqslant 2^{2^{x+y}} \leqslant \mathsf{e}_2(\mathsf{e}_0(x, y)) \in \mathsf{E}_3$$

Next we prove $\mathsf{E}_3 \subseteq \mathsf{E}$. Since E is closed under bounded recursion according to exercise 5, we only need to show $\mathsf{e}_2 \in \mathsf{E}$. Since e_2 is defined by primitive recursion, we need to find an elementary upper bound on e_2:

$$\mathsf{e}_2(x) < 2^{2^{x+1}}$$

This is easily proved by induction on x. For $x = 0$ we obtain $2 < 4$. For the induction step we have

$$\mathsf{e}_2(x+1) = \mathsf{e}_2(x)^2 + 2 \leqslant (2^{2^{x+1}} - 1)^2 + 2 = 2^{2^{x+2}} - 2 \cdot 2^{2^{x+1}} + 3 < 2^{2^{x+2}}$$