universität innsbruck

Computability Theory	26W	LVA 703317
Lecture 2		October 19, 2026

All Solutions

1 (a) We use induction on c. The function $\mathsf{ack}(0, y) = y+1 = \mathsf{s}(y)$ is obviously primitive recursive. Assume that $\mathsf{ack}(c, y)$ is primitive recursive and consider the function $\mathsf{ack}(c+1, y)$. We have

$$\mathsf{ack}(c+1, y) = \begin{cases} \mathsf{ack}(c, 1) & \text{if } y = 0 \\ \mathsf{ack}(c, \mathsf{ack}(c+1, y-1)) & \text{if } y > 0 \end{cases}$$

This is a primitive recursive definition of $\mathsf{ack}(c+1, y)$.

(b) No. Suppose to the contrary that $\mathsf{ack}(x, c)$ is primitive recursive. According to the lemma on slide 17 there exists a constant d such that $\mathsf{ack}(x, c) < \mathsf{ack}(d, x)$ for all $x \in \mathbb{N}$. In particular $\mathsf{ack}(d+2, c) < \mathsf{ack}(d, d+2)$. However,

$$\mathsf{ack}(d+1, c) \geqslant \mathsf{ack}(0, c+d+1) = c+d+2 \geqslant d+2$$

by part 3 of the lemma on slide 12 and thus

$$\mathsf{ack}(d+2, c) \geqslant \mathsf{ack}(d+1, c+1) = \mathsf{ack}(d, \mathsf{ack}(d+1, c)) \geqslant \mathsf{ack}(d, d+2)$$

by part 2 of the lemma on slide 12.

2 We use induction on x. If $x = 0$ then

$$f(x, \vec{y}) = g(\vec{y}) < \mathsf{ack}(c_1, \mathsf{max}\{\vec{y}\}) = \mathsf{ack}(c_1, x + \mathsf{max}\{\vec{y}\}) \leqslant \mathsf{ack}(c', x + \mathsf{max}\{\vec{y}\})$$

In the induction step we assume the statement is true for x, so

$$f(x, \vec{y}) < \mathsf{ack}(c', x + \mathsf{max}\{\vec{y}\}) \leqslant \mathsf{ack}(c'-1, x+1+\mathsf{max}\{\vec{y}\})$$

and thus

$$\begin{aligned} f(x+1, \vec{y}) &= h(f(x, \vec{y}), x, \vec{y}) \\ &< \mathsf{ack}(c_2, \mathsf{max}\{f(x, \vec{y}), x, \vec{y}\}) \\ &\leqslant \mathsf{ack}(c_2, \mathsf{max}\{\mathsf{ack}(c'-1, x+1+\mathsf{max}\{\vec{y}\}), x, \vec{y}\}) \\ &= \mathsf{ack}(c_2, \mathsf{ack}(c'-1, x+1+\mathsf{max}\{\vec{y}\})) \\ &\leqslant \mathsf{ack}(c', \mathsf{ack}(c'-1, x+1+\mathsf{max}\{\vec{y}\})) \\ &= \mathsf{ack}(c', x+1+\mathsf{max}\{\vec{y}\}) \end{aligned}$$

3 Multiplication can be defined by primitive recursion from z and the composition of $+$ and π^3_1, π^3_3. Hence

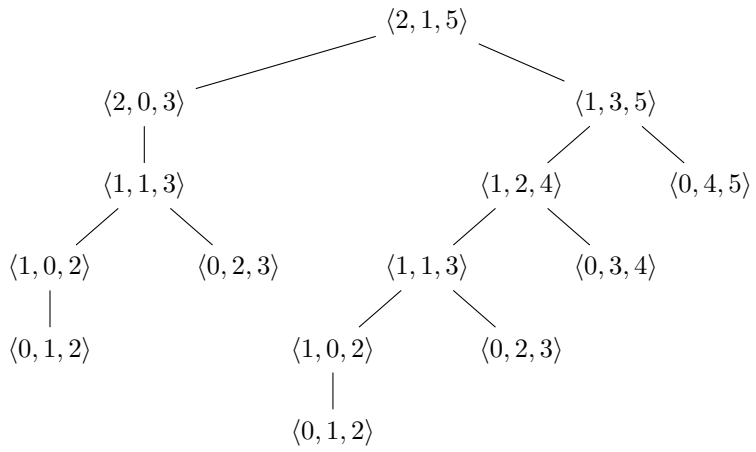
$$\begin{aligned} \ulcorner \times \urcorner &= \langle 4, \ulcorner \mathsf{z} \urcorner, \ulcorner + \circ (\pi^3_1, \pi^3_3) \urcorner \rangle = \langle 4, \langle 0 \rangle, \langle 3, \ulcorner + \urcorner, \ulcorner \pi^3_1 \urcorner, \ulcorner \pi^3_3 \urcorner \rangle \rangle \\ &= \langle 4, \langle 0 \rangle, \langle 3, \langle 4, \langle 2, 1, 1 \rangle, \langle 3, \langle 1 \rangle, \langle 2, 3, 1 \rangle \rangle \rangle, \langle 2, 3, 1 \rangle, \langle 2, 3, 3 \rangle \rangle \rangle \end{aligned}$$

Using the definition on slide 17 and the proof of the lemma on slide 18 of lecture 1, we obtain the following index for exponentiation:

$$\langle 3, \langle 4, \langle 3, \langle 1\rangle, \langle 0\rangle\rangle, \langle 3, \langle 4, \langle 0\rangle, \langle 3, \langle 4, \langle 2,1,1\rangle, \langle 3, \langle 1\rangle, \langle 2,3,1\rangle\rangle\rangle, \langle 2,3,1\rangle, \langle 2,3,3\rangle\rangle\rangle, \langle 2,3,1\rangle, \langle 2,3,3\rangle\rangle\rangle,$$
$$\langle 2,2,2\rangle, \langle 2,2,1\rangle\rangle$$

4 The function f we are looking for is defined by primitive recursion from s and $\mathsf{s} \circ \pi^3_1$. So $g(x) = \mathsf{s}(x)$ and $h(x,y,z) = (\mathsf{s} \circ \pi^3_1)(x,y,z) = \mathsf{s}(x)$ in the primitive recursive definition of f. Hence $f(x,y) = \mathsf{s}(x+y)$ is a binary function that returns the successor of the sum of its arguments.

5 The following tree represents the computation of $\mathsf{ack}(2,1) = 5$:



This tree in encoded in the number

$$\langle\langle 2,1,5\rangle, \langle\langle 2,0,3\rangle, n\rangle, \langle\langle 1,3,5\rangle, \langle\langle 1,2,4\rangle, n, \langle\langle 0,3,4\rangle\rangle\rangle, \langle\langle 0,4,5\rangle\rangle\rangle\rangle$$

with $n = \langle\langle 1,1,3\rangle, \langle\langle 1,0,2\rangle, \langle\langle 0,1,2\rangle\rangle\rangle, \langle\langle 0,2,3\rangle\rangle\rangle$.

6 Given a sequence $a_0, \ldots, a_n$ of natural numbers, we let $c = b!$ for $b = \mathsf{max}\{a_0, \ldots, a_n, n\}$. Define $d_i = 1 + (i+1)c$ for $0 \leqslant i \leqslant n$. We claim that the sequence $d_0, \ldots, d_n$ consists of pairwise relative prime numbers. Suppose e divides both d_i and d_j with $0 \leqslant i < j \leqslant n$. Then e also divides $d_j - d_i = (j-i)c$. Since $0 < j - i \leqslant n$ and $c = b!$ with $b \geqslant n$, c contains $j - i$ as factor. Hence e divides c. It follows that e divides $d_i - (i+1)c = 1$, which proves the claim. According to the Chinese remainder theorem there exists a number d such that $d \mathsf{\ mod\ } d_i = a_i$ for all $0 \leqslant i \leqslant n$. By taking $a = \pi(d,c)$, we obtain

$$\beta(a,i) \;=\; \pi_1(\pi(d,c)) \mathsf{\ mod\ } (1 + (i+1)\pi_2(\pi(d,c))) \;=\; d \mathsf{\ mod\ } d_i \;=\; a_i$$

for $0 \leqslant i \leqslant n$ as desired.