

All Solutions

- 1 (a) The functions $f_1(x) = x \div 2$ and $f_2(x) = 3x + 1$ are primitive recursive. So are the predicates “ x is even” and “ x is odd”:

$$x \text{ is even} \iff 2 \mid x \quad \text{and} \quad x \text{ is odd} \iff \neg(2 \mid x)$$

Hence the primitive recursiveness of f is a consequence of the lemma on [slide 24](#).

- (b) We have $|x - y| = (x \dot{-} y) + (y \dot{-} x) = +(\dot{-}(x, y), \dot{-}(\pi_2^2(x, y), \pi_1^2(x, y)))$ and hence the primitive recursiveness of $|x - y|$ is a consequence of the primitive recursiveness of $+$ and $\dot{-}$.
- (c) We have $g(x) = (\mu y \leq x) [y^2 \leq x \wedge x < (y + 1)^2] = (\mu y \leq x) [x < (y + 1)^2]$.
- (d) The function lcm can be defined by case analysis and bounded minimization:

$$\text{lcm}(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ (\mu i \leq xy) [i > 0 \wedge x \mid i \wedge y \mid i] & \text{if } x, y > 0 \end{cases}$$

- (e) We have $\Pi(x) = \sum_{i \leq x} \chi_{\text{prime}}(x)$.

- 2 (a) We have

$$\pi'(x, y) = \frac{(x + y)(x + y + 1)}{2} + x \tag{1}$$

First we show that $\pi'(x, y) < \pi'(x', y')$ whenever $x + y < x' + y'$. Let $z = x + y$. We have

$$\frac{z(z + 1)}{2} \leq \pi'(x, y) \leq \frac{z(z + 1)}{2} + z < \frac{z(z + 1)}{2} + z + 1 = \frac{(z + 1)(z + 2)}{2} \leq \pi'(x', y')$$

Now if $\pi'(x, y) = \pi'(x', y')$ then $x + y = x' + y'$ and thus $x = x'$ follows from (1). Hence also $y = y'$. It follows that π' is injective. Next we show that π' is surjective. Let $z \in \mathbb{N}$ and consider the largest k such that $\frac{1}{2}k(k + 1) \leq z$. Let $x = z - \frac{1}{2}k(k + 1)$ and $y = k - x$. We have $x + y = k$ and thus $\pi'(x, y) = \frac{1}{2}k(k + 1) + x = z$.

- (b) Define

$$\begin{aligned} \pi'_1(x) &= (\mu y \leq x) (\exists z \leq x) [x = \pi'(y, z)] \\ \pi'_2(x) &= (\mu z \leq x) (\exists y \leq x) [x = \pi'(y, z)] \end{aligned}$$

We have

$$\pi'_1(\pi'(a, b)) = (\mu y \leq \pi'(a, b)) (\exists z \leq \pi'(a, b)) [\pi'(a, b) = \pi'(y, z)]$$

Since π' is injective, $\pi'(a, b) = \pi'(y, z)$ implies $a = y$ and $b = z$. So if we can show $a \leq \pi'_1(\pi'(a, b))$ and $b \leq \pi'_2(\pi'(a, b))$ then $\pi'_1(\pi'(a, b)) = a$ follows. This is easy:

$$\pi'(a, b) = ((a + b)^2 + 3a + b) \div 2 \geq (3a + b^2 + b) \div 2 \geq a, b$$

The proof that $\pi'_2(\pi'(a, b)) = b$ is obtained in the same way.

- 3 (a) First we show by induction on c that the function $h_c(x) = c$ is primitive recursive. If $c = 0$ then $h_c(x) = z(x)$. If $c > 0$ then $h_c(x) = s(h_{c-1}(x))$. The induction hypothesis yields the primitive recursiveness of $h_{c-1}(x)$ and hence the primitive recursiveness of $h_c(x)$ follows by composition. Since $g(x_1, \dots, x_n) = f(\pi_1^n(x_1, \dots, x_n), \dots, \pi_n^n(x_1, \dots, x_n), h_c(\pi_1^n(x_1, \dots, x_n)))$, the function g is primitive recursive.
- (b) Let $g(x, y) = f(x)$. The function g is primitive recursive:

$$\begin{aligned} g(0, y) &= c = h_c(y) \\ g(x+1, y) &= f(x+1) = h(f(x), x) = h'(g(x, y), x, y) \end{aligned}$$

Here $h'(x, y, z) = h(\pi_1^3(x, y, z), \pi_2^3(x, y, z))$ and h_c is the constant function introduced in the previous item. Since $f(x) = g(x, x)$, the primitive recursiveness of f follows from the lemma on [slide 18](#).

4 For instance,

$$\text{sort}(x) = \begin{cases} \langle \rangle & \text{if } \text{len}(x) = 0 \\ \text{insert}((x)_1, \text{sort}(\text{tail}(x))) & \text{if } \text{len}(x) > 0 \end{cases}$$

with

$$\text{tail}(x) = \begin{cases} \langle \rangle & \text{if } \text{len}(x) = 0 \\ p_0^{\text{len}(x) \dot{-} 1} \times \prod_{i=1}^{\text{len}(x) \dot{-} 1} p_i^{(x)_{i+1}} & \text{if } \text{len}(x) > 0 \end{cases}$$

and

$$\text{insert}(x, y) = \begin{cases} \langle x \rangle & \text{if } \text{len}(y) = 0 \\ \langle x \rangle ; y & \text{if } \text{len}(y) > 0 \text{ and } (y)_1 \geq x \\ (y)_1 ; \text{insert}(x, \text{tail}(y)) & \text{if } \text{len}(y) > 0 \text{ and } (y)_1 < x \end{cases}$$

Since it can be shown that $\text{tail}(x) < x$, **sort** and **insert** are defined by course-of-values recursion.