universität innsbruck

26W lecture 6



Automata and Logic

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Outline

1. **Summary of Previous Lecture**
2. **WMSO Definability**
3. **Intermezzo**
4. **WMSO Definability**
5. **MONA**
6. **Further Reading**

Definition

equivalence relation $\equiv_M$ on Σ^* for DFA $M = (Q, \Sigma, \delta, s, F)$ is defined as follows:

$$x \equiv_M y \iff \widehat{\delta}(s, x) = \widehat{\delta}(s, y)$$

Lemmata

- $\equiv_M$ is right congruent: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ for all $a \in \Sigma$ $xa \equiv_M ya$
- $\equiv_M$ refines $L(M)$: for all $x, y \in \Sigma^*$ $x \equiv_M y \implies$ either $x, y \in L(M)$ or $x, y \notin L(M)$
- $\equiv_M$ is of finite index: $\equiv_M$ has finitely many equivalence classes

Definition

Myhill–Nerode relation for $L \subseteq \Sigma^*$ is right congruent equivalence relation of finite index on Σ^* that refines L

Definition

for any set $L \subseteq \Sigma^*$ equivalence relation $\equiv_L$ on Σ^* is defined as follows:

$$x \equiv_L y \iff \text{for all } z \in \Sigma^* \quad \big(xz \in L \iff yz \in L\big)$$

Theorem

1. for every regular set $L \subseteq \Sigma^*$

   there exists one–to–one correspondence (up to isomorphism of automata) between

   - DFAs for L with input alphabet Σ and without inaccessible states
   - Myhill–Nerode relations for L

2. for every set $L \subseteq \Sigma^*$

   L is regular $\iff$ L admits Myhill–Nerode relation $\iff$ $\equiv_L$ is of finite index

3. regular sets are WMSO definable

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

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$\varphi = \exists X.X(x) \to \exists y.x < y \wedge Y(y)$

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$\varphi = \exists X.X(x) \to \exists y.x < y \wedge Y(y)$ $\qquad \mathrm{FV}(\varphi) = (x, Y)$

Definitions

- FV(φ) denotes list of free variables in φ in fixed order with first–order variables preceding second–order ones
- assignment for φ with FV(φ) $= (x_1, \ldots, x_m, X_1, \ldots, X_n)$ is tuple $(i_1, \ldots, i_m, I_1, \ldots, I_n)$ such that $i_1, \ldots, i_m$ are elements of $\mathbb{N}$ and $I_1, \ldots, I_n$ are finite subsets of $\mathbb{N}$

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$\varphi = \exists X.X(x) \to \exists y.x < y \wedge Y(y)$ $\qquad$ $\mathrm{FV}(\varphi) = (x, Y)$

Notation

$(\boldsymbol{i}, \boldsymbol{I})$ for $(i_1, \ldots, i_m, I_1, \ldots, I_n)$ $\qquad$ $\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Example

x_1	0	0	0	1	0	0
X_1	1	0	1	0	1	0
X_2	0	0	0	0	0	0

Example

	0	1	2	3	4	5
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induced assignment

$i_1 = 3$

Example

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x_1	0	0	0	1	0	0	$i_1 = 3$
X_1	1	0	1	0	1	0	$I_1 = \{0, 2, 4\}$
X_2	0	0	0	0	0	0	

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Remark

assignments are identified with strings over $\{0,1\}^{m+n}$ (here $k = m + n$)

$$\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{\ell 1} \\ \vdots & \vdots & & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{\ell k} \end{pmatrix} \approx \begin{pmatrix} a_{11} \\ \vdots \\ a_{1k} \end{pmatrix} \begin{pmatrix} a_{21} \\ \vdots \\ a_{2k} \end{pmatrix} \cdots \begin{pmatrix} a_{\ell 1} \\ \vdots \\ a_{\ell k} \end{pmatrix}$$

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- if $x, y \in (\{0,1\}^{m+n})^*$ induce same assignment then $x = y\mathbf{0}\cdots\mathbf{0}$ or $y = x\mathbf{0}\cdots\mathbf{0}$

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- $\epsilon \in (\{0,1\}^{m+n})^*$ is m–admissible if and only if $m = 0$

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- if $\underline{x} = (\boldsymbol{i}, \boldsymbol{I})$ then $|x| > k$ for all $k \in \{i_1, \ldots, i_m\} \cup I_1 \cup \cdots \cup I_n$

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Lemma

set of m–admissible strings over $\{0,1\}^{m+n}$ is regular

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Question

Consider the following three strings over $\{0,1\}^{2+2}$. Which statements hold ?

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	0	1	2	3
x_1	0	1	0	0
x_2	0	0	1	0
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A the first string is 2−admissible

B the second and third string induce the same assignment

C $X_2(x_1)$ is satisfied by the first string's induced assignment

D the third string induces the assignment $i_1 = 1$, $i_2 = 2$, $I_1 = \{0\}$, $I_2 = \{1,2\}$

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Proof (construction)

define DFA $\mathcal{A}_{m,n} = (Q, \Sigma, \delta, s, F)$

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2. $Q = 2^{\{1,\ldots,m\}} \cup \{\perp\}$

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② $Q = 2^{\{1,\ldots,m\}} \cup \{\bot\}$

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⑤ $\delta(q,a) = \begin{cases} q - I & \text{if } q \subseteq \{1,\ldots,m\} \text{ and } I \subseteq q \\ \bot & \text{if } q \subseteq \{1,\ldots,m\} \text{ and } I \not\subseteq q \\ \bot & \text{if } q = \bot \end{cases}$

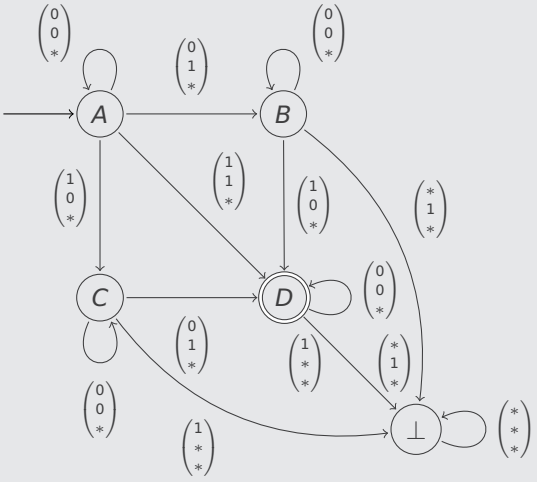
with $I = \{i \in \{1,\ldots,m\} \mid a_i = 1\}$

Example

DFA $\mathcal{A}_{2,1}$

Example

DFA $\mathcal{A}_{2,1}$ with $A = \{1,2\}$, $B = \{1\}$, $C = \{2\}$, $D = \varnothing$



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Theorem

$L_a(\varphi)$ is regular for every WMSO formula φ

Proof

induction on φ

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- $\varphi = \bot$
- $\varphi = x < y$
- $\varphi = X(x)$
- $\varphi = \neg\psi$
- $\varphi = \varphi_1 \vee \varphi_2$

Proof

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- $\varphi = X(x) \quad \Longrightarrow \quad L_a(\varphi) = \left[\binom{0}{0} + \binom{0}{1}\right]^*\binom{1}{1}\left[\binom{0}{0} + \binom{0}{1}\right]^*$
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- $\varphi = X(x) \implies L_a(\varphi) = \left[\binom{0}{0} + \binom{0}{1}\right]^*\binom{1}{1}\left[\binom{0}{0} + \binom{0}{1}\right]^*$
- $\varphi = \neg\psi \implies L_a(\psi)$ is regular

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- $\varphi = \varphi_1 \vee \varphi_2$

  $L_a(\varphi_1)$ and $L_a(\varphi_2)$ are regular but may be defined over different alphabets because φ_1 and φ_2 may have less free variables than φ

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  $\Longrightarrow \quad L_a(\varphi) = \sim L_a(\psi) \cap L(\mathcal{A}_{m,n})$ is regular (for suitable m and n)
- $\varphi = \varphi_1 \vee \varphi_2$ with $\mathrm{FV}(\varphi) = (x_1, \ldots, x_m, X_1, \ldots, X_n)$

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Example

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Definition

homomorphism

$$\mathrm{drop}_i \colon (\{0,1\}^k)^* \to (\{0,1\}^{k-1})^*$$

is defined for $1 \leqslant i \leqslant k$ by dropping i–th component from vectors in $\{0,1\}^k$

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homomorphism

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Lemmata

- $A \subseteq (\{0,1\}^k)^*$ is regular $\quad \Longrightarrow \quad \mathrm{drop}_i(A) \subseteq (\{0,1\}^{k-1})^*$ is regular

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Lemmata

- $A \subseteq (\{0,1\}^k)^*$ is regular $\implies$ $\mathrm{drop}_i(A) \subseteq (\{0,1\}^{k-1})^*$ is regular
- $B \subseteq (\{0,1\}^{k-1})^*$ is regular $\implies$ $\mathrm{drop}_i^{-1}(B) \subseteq (\{0,1\}^k)^*$ is regular

Example

$\varphi = x < y \vee X(x)$ with $\mathrm{FV}(\varphi) = (x, y, X)$

- $L_a(x < y) = \binom{0}{0}^* \binom{1}{0} \binom{0}{0}^* \binom{0}{1} \binom{0}{0}^*$
- $L_a(X(x)) = \left[\binom{0}{0} + \binom{0}{1}\right]^* \binom{1}{1} \left[\binom{0}{0} + \binom{0}{1}\right]^*$
- $L_1 = \mathrm{drop}_3^{-1}\left(\binom{0}{0}^* \binom{1}{0} \binom{0}{0}^* \binom{0}{1} \binom{0}{0}^*\right)$

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- $L_2 = \mathrm{drop}_2^{-1}\left(\left[\binom{0}{0} + \binom{0}{1}\right]^* \binom{1}{1} \left[\binom{0}{0} + \binom{0}{1}\right]^*\right)$

Example

$\varphi = x < y \vee X(x)$ with $\mathrm{FV}(\varphi) = (x, y, X)$

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Example

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- $L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{2,1})$

Proof

induction on φ

- $\varphi = \bot \quad \Longrightarrow \quad L_a(\varphi) = \varnothing$
- $\varphi = x < y \quad \Longrightarrow \quad L_a(\varphi) = \binom{0}{0}^*\binom{1}{0}\binom{0}{0}^*\binom{0}{1}\binom{0}{0}^*$ or $L_a(\varphi) = \binom{0}{0}^*\binom{0}{1}\binom{0}{0}^*\binom{1}{0}\binom{0}{0}^*$
- $\varphi = X(x) \quad \Longrightarrow \quad L_a(\varphi) = \left[\binom{0}{0} + \binom{0}{1}\right]^*\binom{1}{1}\left[\binom{0}{0} + \binom{0}{1}\right]^*$
- $\varphi = \neg\psi \quad \Longrightarrow \quad L_a(\psi)$ is regular $\quad \Longrightarrow \quad \sim L_a(\psi)$ is regular

  $\Longrightarrow \quad L_a(\varphi) = \sim L_a(\psi) \cap L(\mathcal{A}_{m,n})$ is regular (for suitable m and n)
- $\varphi = \varphi_1 \vee \varphi_2$ with $\mathrm{FV}(\varphi) = (x_1, \ldots, x_m, X_1, \ldots, X_n)$

  $L_a(\varphi_1)$ and $L_a(\varphi_2)$ are regular but may be defined over different alphabets because φ_1 and φ_2 may have less free variables than φ

  applications of inverse homomorphism drop_i^{-1} to $L_a(\varphi_1)$ and $L_a(\varphi_2)$ yield regular sets $L_1, L_2 \subseteq (\{0,1\}^{m+n})^*$ such that $L_a(\varphi) = (L_1 \cup L_2) \cap L(\mathcal{A}_{m,n})$

Proof (cont'd)

- $\varphi = \exists x.\, \psi$

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Proof (cont'd)

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Example

$\varphi = \exists X.\psi$ with $\psi = X(x) \wedge \exists y.x < y \wedge X(y)$

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- $L_a(\psi) = \left[\binom{0}{0} + \binom{0}{1}\right]^* \binom{1}{1} \left[\binom{0}{0} + \binom{0}{1}\right]^* \binom{0}{1} \left[\binom{0}{0} + \binom{0}{1}\right]^*$

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- $\text{drop}_2(L_a(\psi)) = (0+0)^*1(0+0)^*0(0+0)^*$

Proof (cont'd)

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- $\mathrm{drop}_2(L_a(\psi)) = (0+0)^*1(0+0)^*0(0+0)^* = 0^*100^* \neq 0^*10^* = L_a(\varphi)$

Definition

$\mathrm{stz}(A) = \{x \mid x\mathbf{0}\cdots\mathbf{0} \in A\}$ for $A \subseteq (\{0,1\}^{m+n})^*$

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Proof

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- $L(M') = \mathrm{stz}(A)$

Proof (cont'd)

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- $\text{drop}_2(L_a(\psi)) = (0+0)^*1(0+0)^*0(0+0)^* = 0^*100^* \neq 0^*10^* = L_a(\varphi)$

Final Task

transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with free variables in $\{P_a \mid a \in \Sigma\}$

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Procedure

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$$\{0^k 1 0^l \mid k + 1 + l = |\Sigma|\}^* \quad \text{is regular}$$

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1. eliminate assignments which do not correspond to string in Σ^*
2. map strings in 0^*10^* to elements of Σ

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$$L_a(\varphi) \cap \{0^k10^l \mid k+1+l = |\Sigma|\}^* \quad \text{is regular}$$

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transform $L_a(\varphi)$ into $L(\varphi)$ for WMSO formula φ with free variables in $\{P_a \mid a \in \Sigma\}$ using regularity preserving operations

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1. eliminate assignments which do not correspond to string in Σ^*
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$L_a(\varphi) \cap \{0^k10^l \mid k+1+l = |\Sigma|\}^*$ is regular

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$L(\varphi) = h(L_a(\varphi) \cap \{0^k10^l \mid k+1+l = |\Sigma|\}^*)$ is regular

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Corollary

WMSO definable sets are regular

Outline

1. Summary of Previous Lecture
2. WMSO Definability
3. Intermezzo
4. WMSO Definability
5. **MONA**
6. Further Reading

MONA

- MONA is state–of–the–art tool that implements decision procedures for WS1S and WS2S

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Demo

`https://www.brics.dk/mona/`

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5. MONA
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Ebbinghaus, Flum and Thomas

- Section 10.9 of Einführung in die mathematische Logik (Springer Spektrum 2018)

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Klarlund and Møller

- Section 3 of MONA Version 1.4 User Manual (2001)

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Important Concepts

- drop_i
- $L_a(\varphi)$
- $m-$admissible
- MONA
- stz
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homework for November 27