universität innsbruck

26W lecture 5



Automata and Logic

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Outline

1. **Summary of Previous Lecture**
2. **WMSO Definability**
3. **Intermezzo**
4. **Myhill–Nerode Relations**
5. **Further Reading**

Definitions

DFA $M = (Q, \Sigma, \delta, s, F)$

- state p is inaccessible if $\widehat{\delta}(s, x) \neq p$ for all $x \in \Sigma^*$
- states p and q are indistinguishable ($p \approx q$) if $\widehat{\delta}(p, x) \in F \iff \widehat{\delta}(q, x) \in F$ for all $x \in \Sigma^*$

Minimization Algorithm

DFA $M = (Q, \Sigma, \delta, s, F)$

1. remove inaccessible states
2. determine which states are indistinguishable by marking algorithm
3. collapse indistinguishable states

Marking Algorithm

given DFA $M = (Q, \Sigma, \delta, s, F)$ without inaccessible states

① tabulate all unordered pairs $\{p, q\}$ with $p, q \in Q$, initially unmarked

② mark $\{p, q\}$ if $p \in F$ and $q \notin F$ or $p \notin F$ and $q \in F$

③ repeat until no change: mark $\{p, q\}$ if $\{\delta(p, a), \delta(q, a)\}$ is marked for some $a \in \Sigma$

Lemmata

- $p \approx q \iff \{p, q\}$ is unmarked
- $\approx$ is equivalence relation on Q

Notation

$[p]_{\approx} = \{q \in Q \mid p \approx q\}$ denotes equivalence class of p

Definition (Collapsing Indistinguishable States)

DFA $M/\approx$ is defined as $(Q', \Sigma, \delta', s', F')$ with

- $Q' = \{[p]_\approx \mid p \in Q\}$
- $\delta'([p]_\approx, a) = [\delta(p,a)]_\approx$ well–defined: $p \approx q \implies \delta(p,a) \approx \delta(q,a)$
- $s' = [s]_\approx$
- $F' = \{[p]_\approx \mid p \in F\}$

Theorem

$L(M/\approx) = L(M)$

Question

is $M/\approx$ minimum–state DFA for $L(M)$?

Definitions

- first–order variables $V_1 = \{x, y, \ldots\}$ ranging over natural numbers
- second–order variables $V_2 = \{X, Y, \ldots\}$ ranging over finite sets of natural numbers
- formulas of weak monadic second–order logic (WMSO)

$$\varphi ::= \bot \mid x < y \mid X(x) \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \exists x.\varphi \mid \exists X.\varphi$$

with $x, y \in V_1$ and $X \in V_2$

Abbreviations

$$\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi) \qquad \varphi \to \psi := \neg\varphi \vee \psi$$

$$\forall x.\varphi := \neg\exists x.\neg\varphi \qquad \forall X.\varphi := \neg\exists X.\neg\varphi$$

$$x \leqslant y := \neg(y < x) \qquad x = y := x \leqslant y \wedge y \leqslant x$$

$$\top := \neg\bot \qquad x = 0 := \neg\exists y.\, y < x$$

$$X(0) := \exists x.X(x) \wedge x = 0 \qquad z = y + 1 := y < z \wedge \neg\exists x.\, y < x \wedge x < z$$

Remarks

- $X(x)$ represents $x \in X$
- MSO is WMSO without restriction to finite sets

Definitions

- assignment α is mapping from variables $x \in V_1$ to $\mathbb{N}$ and $X \in V_2$ to finite subsets of $\mathbb{N}$
- assignment α satisfies formula φ ($\alpha \vDash \varphi$):

$$
\begin{aligned}
&\alpha \nvDash \bot \\
&\alpha \vDash x < y && \iff && \alpha(x) < \alpha(y) \\
&\alpha \vDash X(x) && \iff && \alpha(x) \in \alpha(X) \\
&\alpha \vDash \neg\varphi && \iff && \alpha \nvDash \varphi \\
&\alpha \vDash \varphi_1 \vee \varphi_2 && \iff && \alpha \vDash \varphi_1 \text{ or } \alpha \vDash \varphi_2 \\
&\alpha \vDash \exists x.\, \varphi && \iff && \alpha[x \mapsto n] \vDash \varphi \qquad \text{for some } n \in \mathbb{N} \\
&\alpha \vDash \exists X.\, \varphi && \iff && \alpha[X \mapsto N] \vDash \varphi \qquad \text{for some finite subset } N \subset \mathbb{N}
\end{aligned}
$$

Definitions

- formula φ is satisfiable if $\alpha \vDash \varphi$ for some assignment α
- formula φ is valid if $\alpha \vDash \varphi$ for all assignments α
- model of formula φ is assignment α such that $\alpha \vDash \varphi$
- size of model α is smallest n such that
  1. $\alpha(x) < n$ for $x \in V_1$
  2. $\alpha(X) \subseteq \{0, \ldots, n-1\}$ for $X \in V_2$

Definition

given alphabet Σ and string $x = a_0 \cdots a_{n-1} \in \Sigma^*$

- second–order variables $V_2 = \{P_a \mid a \in \Sigma\}$
- $\alpha_x(P_a) = \{i < n \mid x_i = a\}$

Notation

$\underline{x}$ for α_x

Definitions

- given alphabet Σ and WMSO formula φ with free variables (exclusively) in $\{P_a \mid a \in \Sigma\}$

$$L(\varphi) = \{x \in \Sigma^* \mid \underline{x} \vDash \varphi\}$$

- set $A \subseteq \Sigma^*$ is WMSO definable if $A = L(\varphi)$ for some WMSO formula φ

Theorem

set $A \subseteq \Sigma^*$ is regular if and only if A is WMSO definable

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

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Definitions

DFA $M = (Q, \Sigma, \delta, s, F)$

- run of M on input $x = a_1 \cdots a_n \in \Sigma^*$ is sequence $q_0, \ldots, q_n$ of states such that

$$s = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n$$

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- run $q_0, \ldots, q_n$ is accepting if $q_n \in F$

Proof ($\Longrightarrow$)

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- DFA $M = (Q, \Sigma, \delta, s, F)$ with $Q = \{q_1, \ldots, q_m\}$
- second–order variables $X_{q_1}, \ldots, X_{q_m}$

Remarks

- $X_q = \{i \mid \widehat{\delta}(s, a_1 \cdots a_i) = q\}$ for input $a_1 \cdots a_n \in \Sigma^*$

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Example

- run $s \xrightarrow{a} p \xrightarrow{a} q \xrightarrow{b} p$

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Example

- run $s \xrightarrow{a} p \xrightarrow{a} q \xrightarrow{b} p$
- assignment

$$P_a = \{0, 1\} \qquad P_b = \{2\}$$

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Proof ($\Longrightarrow$)

- DFA $M = (Q, \Sigma, \delta, s, F)$ with $Q = \{q_1, \ldots, q_m\}$
- second–order variables $X_{q_1}, \ldots, X_{q_m}$ to encode accepting runs of M as WMSO formula φ_M

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$$\varphi_M := \exists X_{q_1}. \cdots \exists X_{q_m}. \exists \ell. \bigwedge_{a \in \Sigma} \neg P_a(\ell) \wedge \left(\forall x. \bigwedge_{a \in \Sigma} \neg P_a(x) \rightarrow \ell \leqslant x \right) \wedge \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4$$

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- $X_q = \{i \mid \widehat{\delta}(s, a_1 \cdots a_i) = q\}$ for input $a_1 \cdots a_n \in \Sigma^*$
- ℓ denotes length n of input

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$$\psi_1 := X_s(0)$$

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$$\psi_3 := \forall x. x < \ell \rightarrow \bigvee_{a \in \Sigma,\, q \in Q} X_q(x) \wedge P_a(x) \wedge \exists y. y = x + 1 \wedge X_{\delta(q,a)}(y)$$

Proof ($\Longrightarrow$)

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$$\psi_4 := \bigvee_{q \in F} X_q(\ell)$$

Remarks

- $X_q = \{i \mid \widehat{\delta}(s, a_1 \cdots a_i) = q\}$ for input $a_1 \cdots a_n \in \Sigma^*$
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Example

- run $s \xrightarrow{a} p \xrightarrow{a} q \xrightarrow{b} p$
- assignment

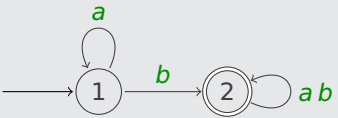
$$P_a = \{0, 1\} \qquad P_b = \{2\} \qquad X_s = \{0\} \qquad X_p = \{1, 3\} \qquad X_q = \{2\}$$

Proof ($\Longrightarrow$, cont'd)

- $L(\varphi_M) = L(M)$

Example

- DFA M



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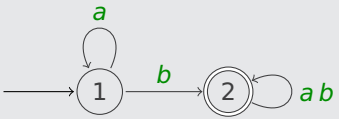

- WMSO formula φ_M

$$\exists X_1.\, \exists X_2.\, \exists \ell.\, \neg P_a(\ell) \wedge \neg P_b(\ell) \wedge \big(\forall x.\, \neg P_a(x) \wedge \neg P_b(x) \rightarrow \ell \leqslant x\big) \wedge \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4$$

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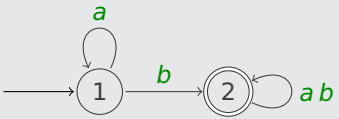

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with

$\psi_1 = X_1(0)$

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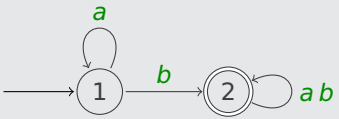

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Example

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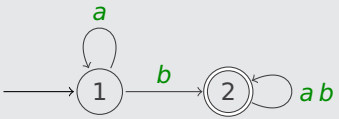

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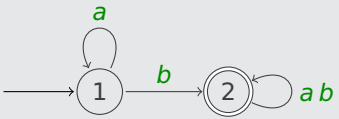

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Example

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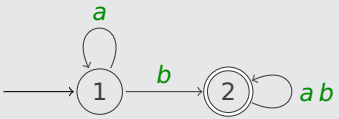

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Example

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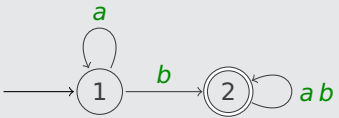

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$$\psi_4 = X_2(\ell)$$

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Question

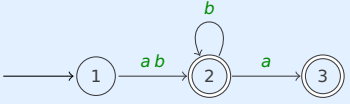
Consider the language encoded by the following WMSO formula over $\Sigma = \{a, b\}$:

$$\varphi = \exists \ell. \neg P_a(\ell) \wedge \neg P_b(\ell) \wedge \big(\forall x. \neg P_a(x) \wedge \neg P_b(x) \rightarrow \ell \leqslant x\big) \\ \wedge \big(\forall x. P_a(x) \rightarrow x = 0 \vee \ell = x + 1\big)$$

Which of the following statements hold ?

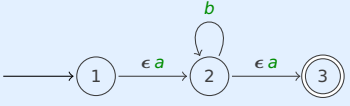
A $L(\varphi) = \Sigma^*$

B $L(\varphi) = L(N)$ for the NFA N:



C $L(\varphi) = L(ab^* + ab^*a + b^* + b^*a)$

D $L(\varphi) = L(N')$ for the NFA$_\epsilon$ N':



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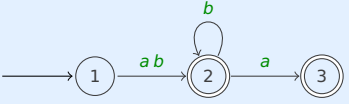
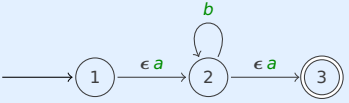


- ☑ $L(\varphi) = L(ab^* + ab^*a + b^* + b^*a)$
- ☑ $L(\varphi) = L(N')$ for the NFA$_\epsilon$ N':



Outline

1. Summary of Previous Lecture
2. WMSO Definability
3. Intermezzo
4. **Myhill–Nerode Relations**
5. Further Reading

Definition

equivalence relation $\equiv_M$ on Σ^* for DFA $M = (Q, \Sigma, \delta, s, F)$ is defined as follows:

$$x \equiv_M y \quad \iff \quad \widehat{\delta}(s, x) = \widehat{\delta}(s, y)$$

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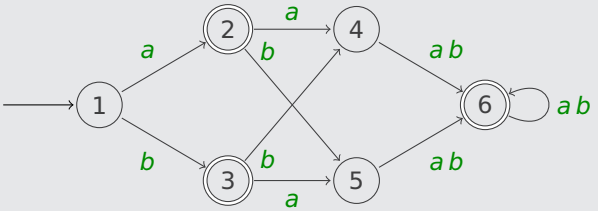
Definition

Myhill–Nerode relation for $L \subseteq \Sigma^*$ is right congruent equivalence relation of finite index on Σ^* that refines L

Example

M



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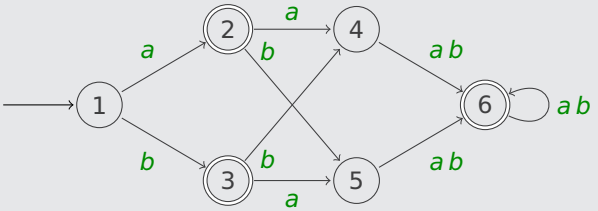

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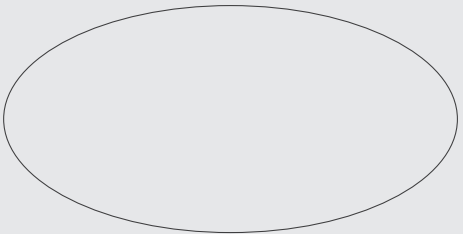

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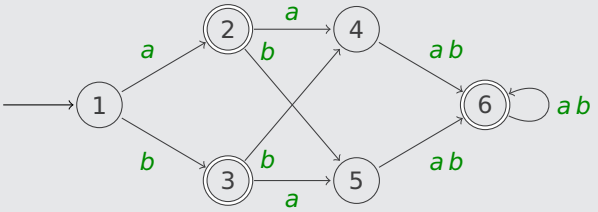

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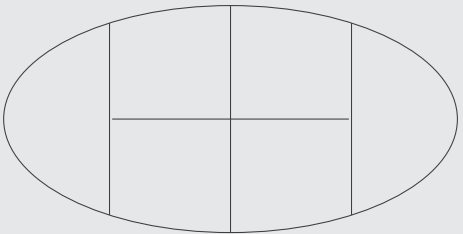

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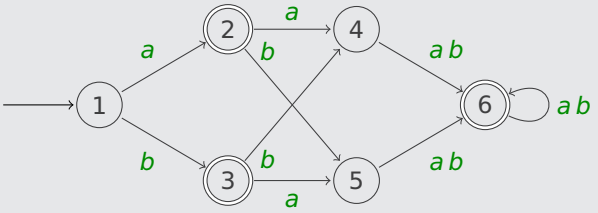

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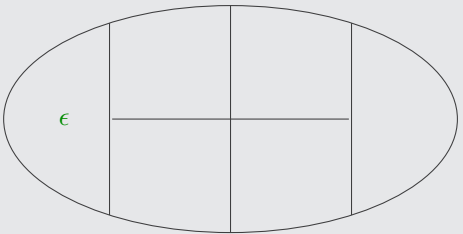

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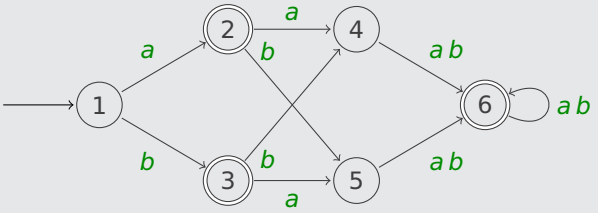

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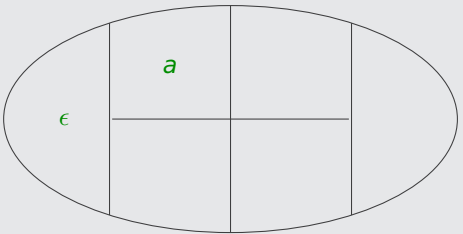

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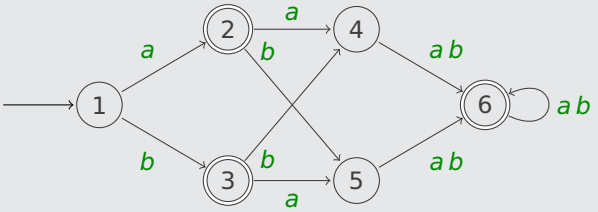

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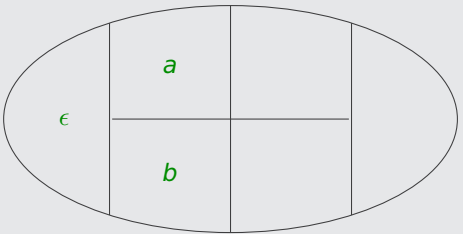

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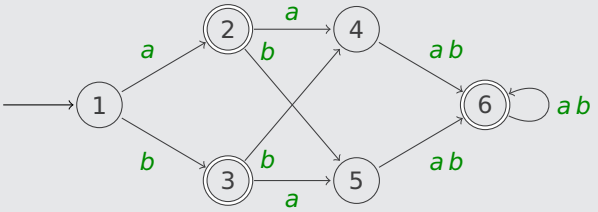

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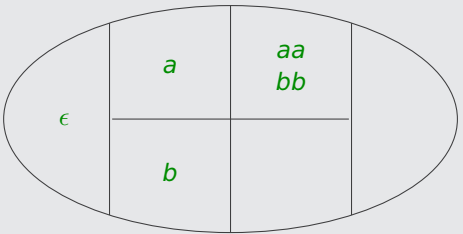

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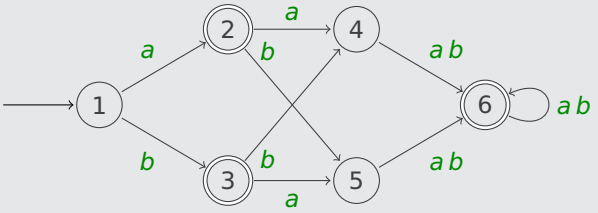

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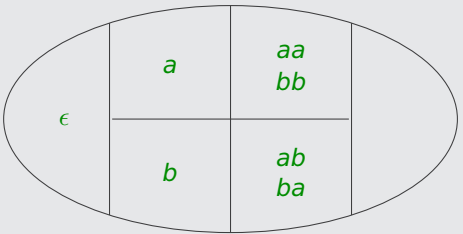

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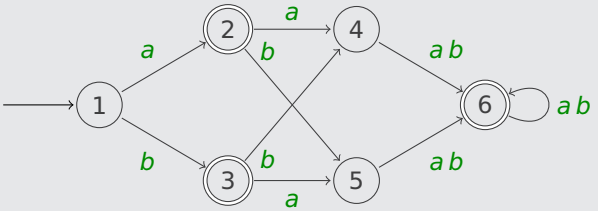

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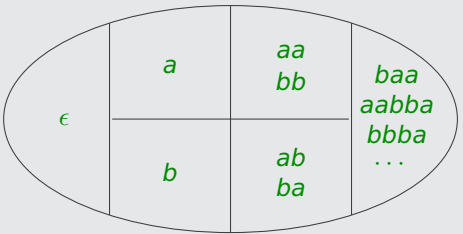

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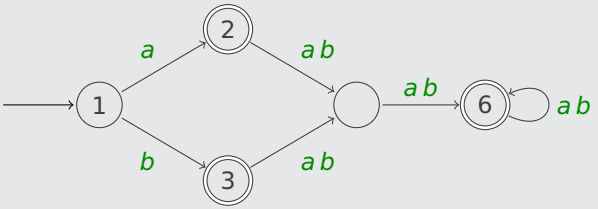

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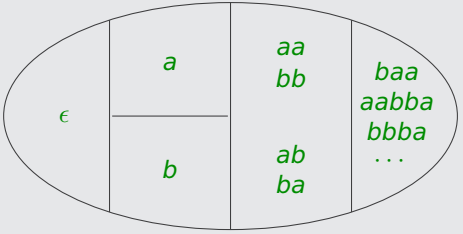

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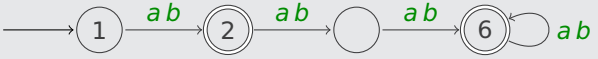

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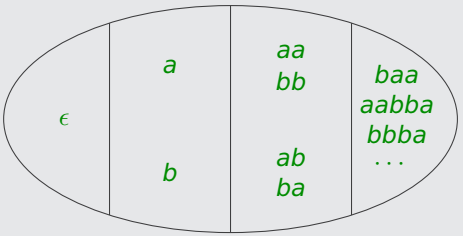

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given Myhill–Nerode relation $\equiv$ for set $L \subseteq \Sigma^*$ DFA $M_{\equiv}$ is defined as $(Q, \Sigma, \delta, s, F)$ with

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2. $x \in L \iff [x]_\equiv \in F$

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Theorem

$L(M_{\equiv}) = L$

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Proof

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Corollary

if L admits Myhill–Nerode relation then L is regular

Theorem

two mappings (for $L \subseteq \Sigma^*$)

- $D \mapsto \equiv_D$ from DFAs for L to Myhill–Nerode relations for L
- $\approx \mapsto M_\approx$ from Myhill–Nerode relations for L to DFAs for L

are each others inverse (up to isomorphism of automata)

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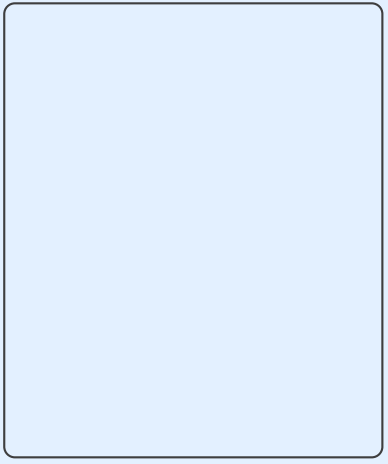
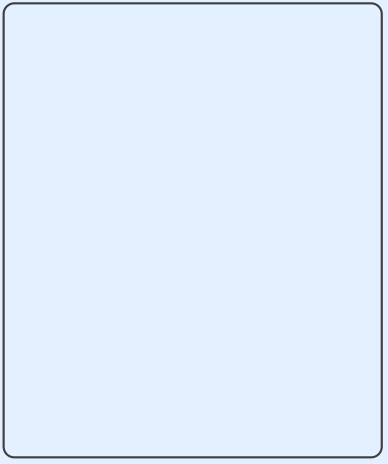
for regular $L \subseteq \Sigma^*$



DFAs for L



Myhill–Nerode relations for L

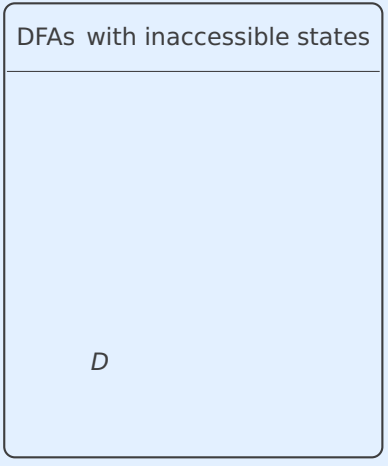
for regular $L \subseteq \Sigma^*$

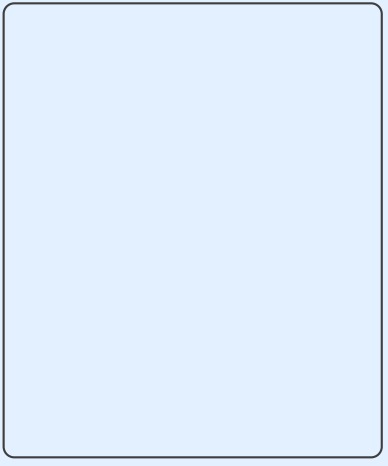
DFAs with inaccessible states

DFAs for L

Myhill–Nerode relations for L

for regular $L \subseteq \Sigma^*$





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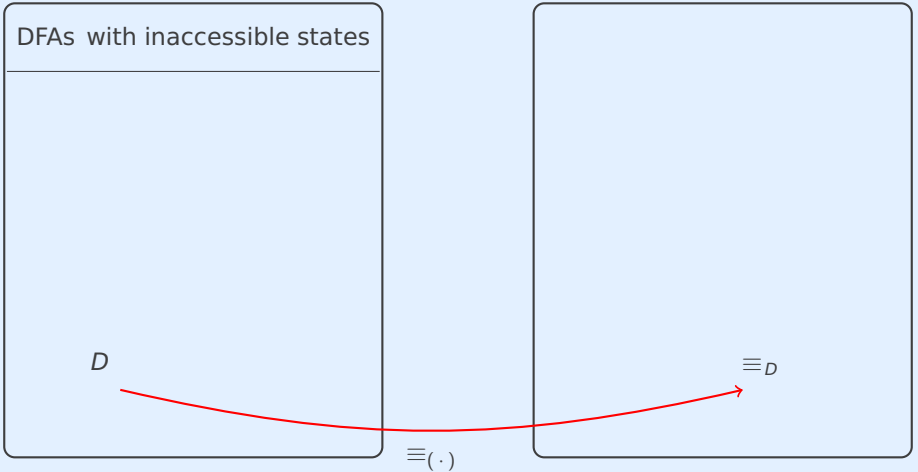

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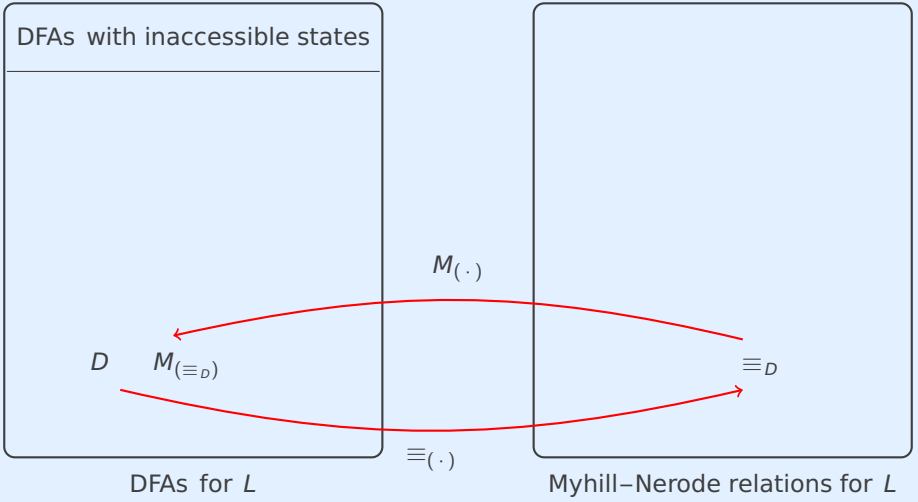

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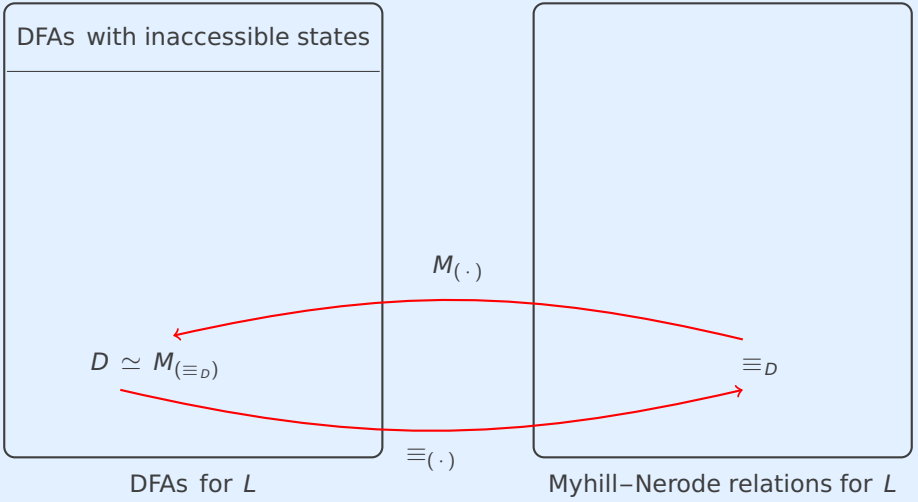

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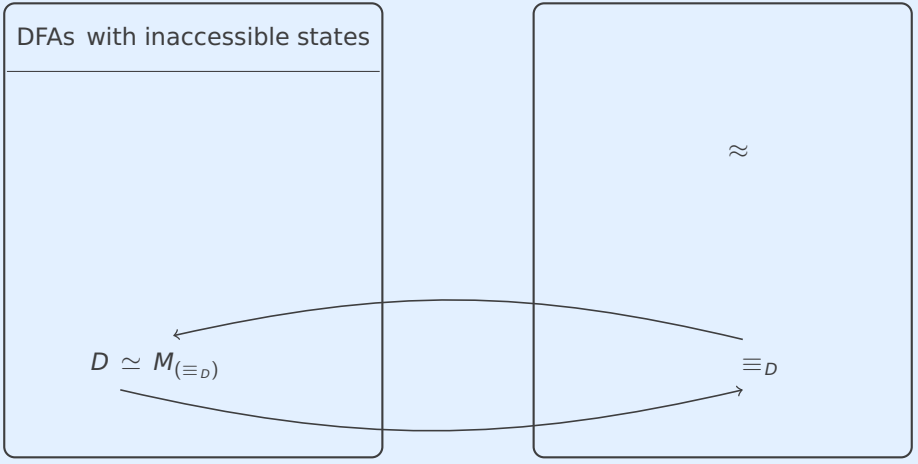

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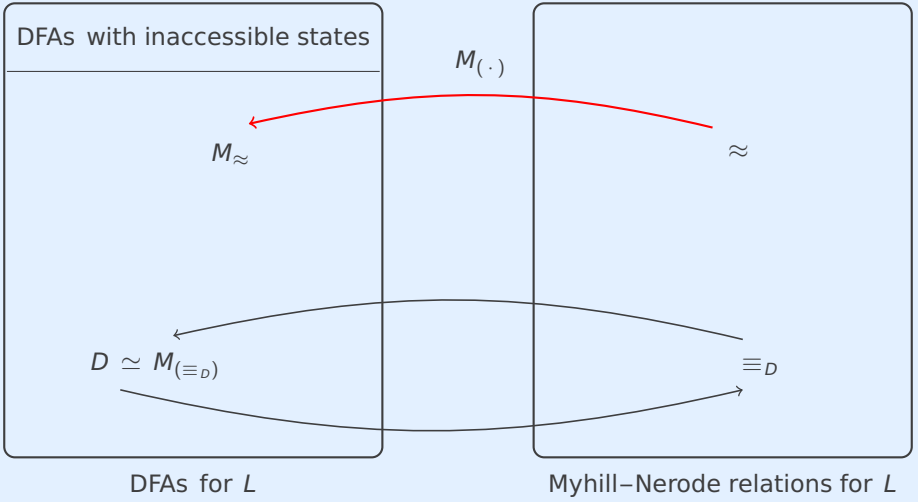

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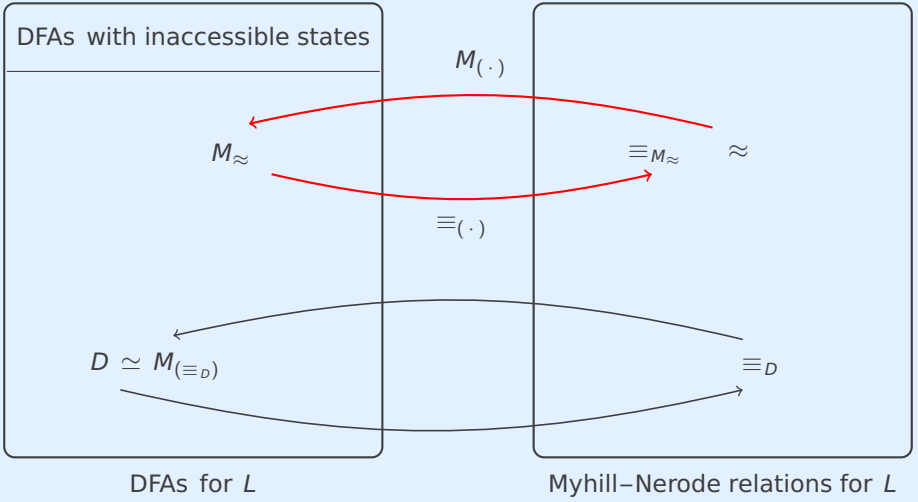

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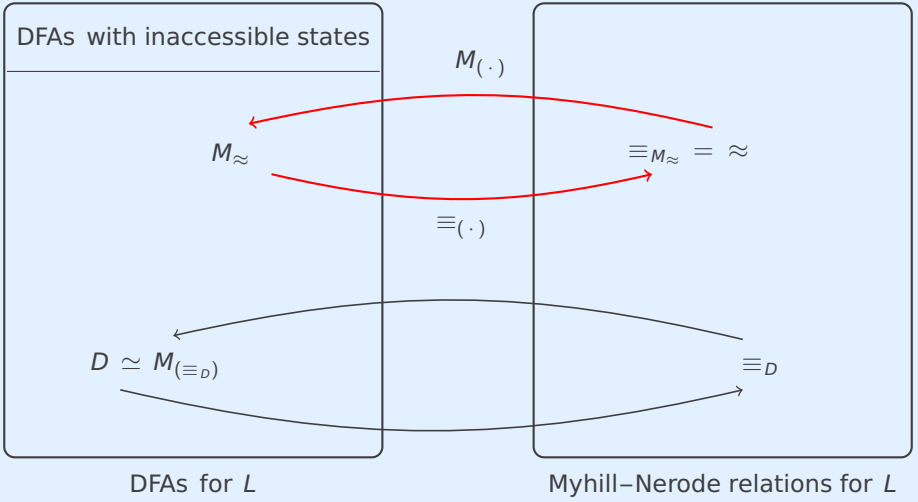

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$$\text{for all } x, y \in \Sigma^* \quad x \sim y \implies x \equiv_L y$$

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$$\text{for all } x, y \in \Sigma^* \quad x \sim y \implies x \equiv_L y$$

$\equiv_L$ has fewest equivalence classes

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following statements are equivalent for any set $L \subseteq \Sigma^*$:

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Corollary

for every regular set L $\quad M_{(\equiv_L)}$ is minimum–state DFA for L

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- $\equiv_D$ does not refine D ↯

Outline

1. Summary of Previous Lecture
2. WMSO Definability
3. Intermezzo
4. Myhill–Nerode Relations
5. **Further Reading**

Kozen

- Lectures 13 – 16

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homework for November 20