universität innsbruck

26W lecture 4



Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Outline

1. **Summary of Previous Lecture**
2. **Minimization**
3. **Intermezzo**
4. **Weak Monadic Second–Order Logic**
5. **Further Reading**

Definitions

- regular expression α over alphabet Σ:

$$a \in \Sigma \qquad \epsilon \qquad \varnothing \qquad \beta + \gamma \qquad \beta\gamma \qquad \beta^*$$

- set of strings $L(\alpha) \subseteq \Sigma^*$ matched by regular expression α:

$$L(a) = \{a\} \qquad L(\varnothing) = \varnothing \qquad L(\beta\gamma) = L(\beta)L(\gamma)$$
$$L(\epsilon) = \{\epsilon\} \qquad L(\beta + \gamma) = L(\beta) \cup L(\gamma) \qquad L(\beta^*) = L(\beta)^*$$

- regular expressions α and β are equivalent $(\alpha \equiv \beta)$ if $L(\alpha) = L(\beta)$

Theorem

finite automata and regular expressions are equivalent:

$$\text{for all } A \subseteq \Sigma^* \quad A \text{ is regular} \iff A = L(\alpha) \text{ for some regular expression } \alpha$$

Definitions

- homomorphism is mapping $h\colon \Sigma^* \to \Gamma^*$ such that $h(\epsilon) = \epsilon$ and $h(xy) = h(x)h(y)$
- if $A \subseteq \Sigma^*$ then $h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$ "image of A under h"
- if $B \subseteq \Gamma^*$ then $h^{-1}(B) = \{x \mid h(x) \in B\} \subseteq \Sigma^*$ "preimage of B under h"

Theorem

regular sets are effectively closed under homomorphic image and preimage

Theorem

problems

instance: DFA M and string x	instance: DFA M	instance: DFAs M and N
question: $x \in L(M)$?	question: $L(M) = \varnothing$?	question: $L(M) = L(N)$?

are decidable

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

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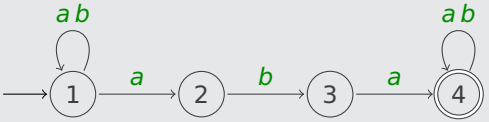
Example 1

NFA



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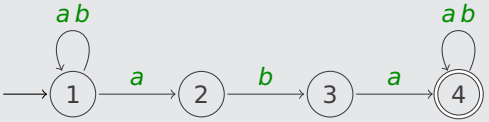

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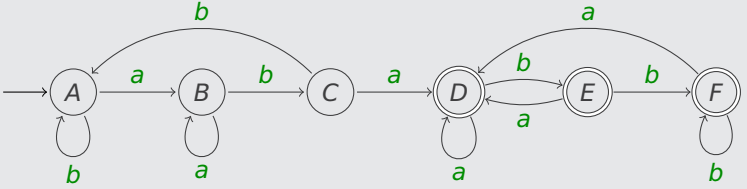
$B = \{1,2\}$ $D = \{1,2,4\}$ $F = \{1,4\}$

DFA



Example ❶

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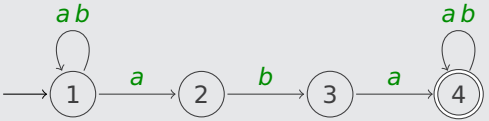

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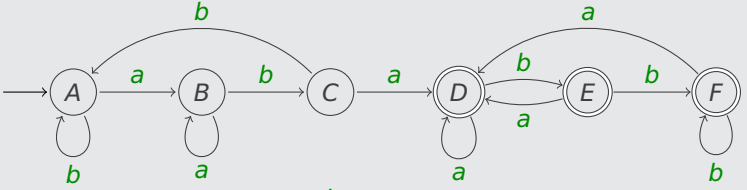
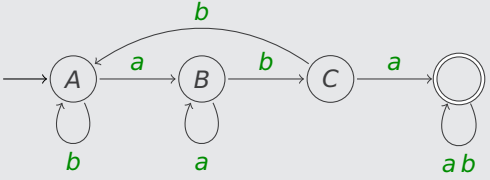


not minimal:



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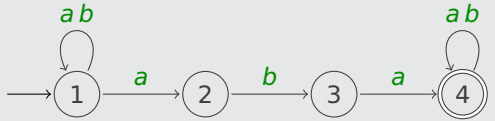

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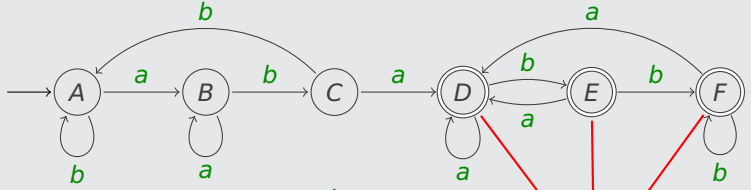

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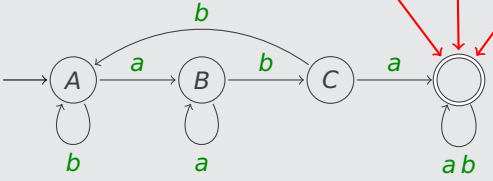

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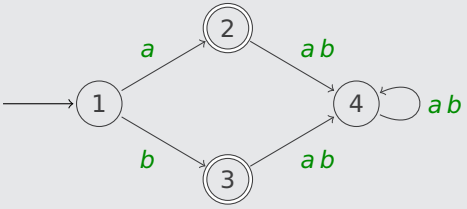

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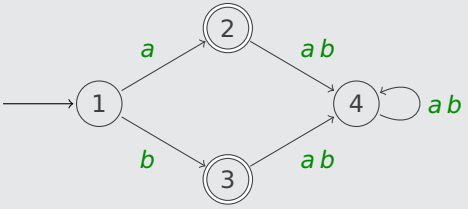

not minimal: states 2 and 3 can be collapsed



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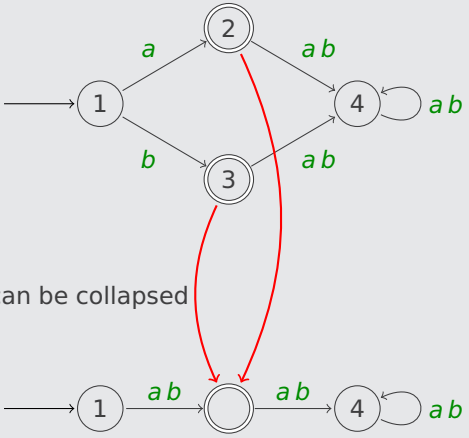

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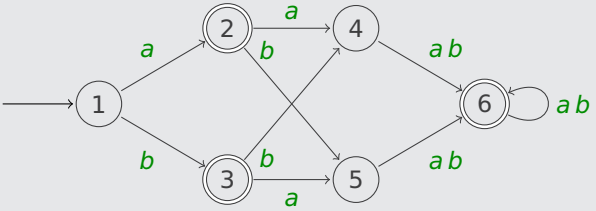

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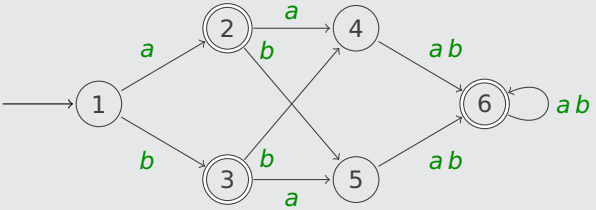

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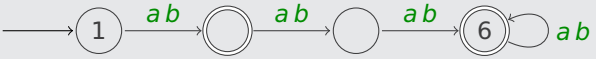

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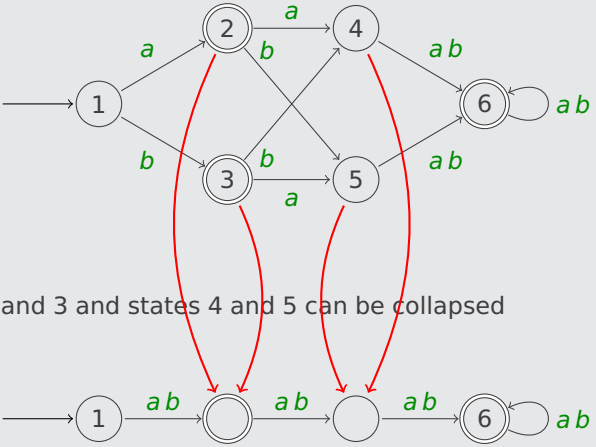

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- state p is inaccessible if $\widehat{\delta}(s, x) \neq p$ for all $x \in \Sigma^*$

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Minimization Algorithm

DFA $M = (Q, \Sigma, \delta, s, F)$

1. remove inaccessible states
2. for every two different states determine whether they are distinguishable (marking)
3. collapse indistinguishable states

Marking Algorithm

given DFA $M = (Q, \Sigma, \delta, s, F)$ without inaccessible states

① tabulate all unordered pairs $\{p, q\}$ with $p, q \in Q$, initially unmarked

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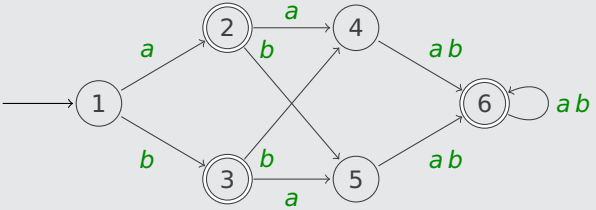
Lemma

$p \approx q \iff \{p, q\}$ is unmarked

Example



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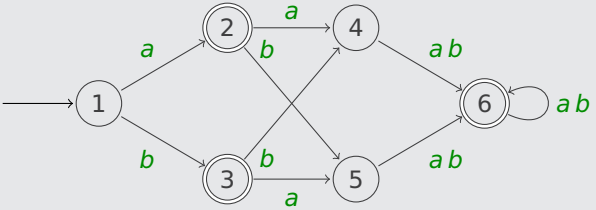
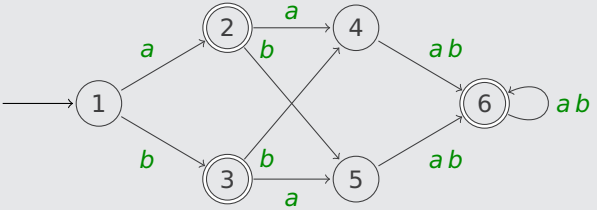


1
✓ 2
✓ 3
✓ ✓ 4
✓ ✓ 5
✓ ✓ ✓ 6

1. final / nonfinal states are distinguishable

Example



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2. $\{2,6\} \xrightarrow{a} \{4,6\}$ $\{3,6\} \xrightarrow{a} \{5,6\}$

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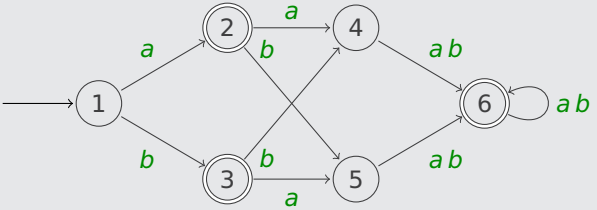

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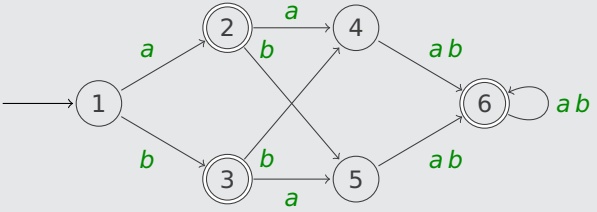

1					
✓	2				
✓		3			
✓	✓	✓	4		
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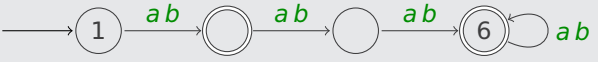
collapse states 2 and 3 and states 4 and 5:



Definition

states p and q of DFA $M = (Q, \Sigma, \delta, s, F)$ are indistinguishable ($p \approx q$) if for all $x \in \Sigma^*$

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Notation

$[p]_{\approx} = \{q \in Q \mid p \approx q\}$ denotes equivalence class of p

Definition (Collapsing Indistinguishable States)

DFA $M/\approx$ is defined as $(Q', \Sigma, \delta', s', F')$ with

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for all $p \in Q$

Theorem

$L(M/\approx) = L(M)$

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Question

is $M/\approx$ minimum–state DFA for $L(M)$?

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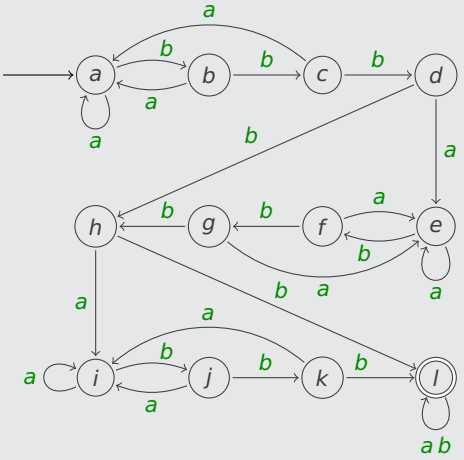
Lemma

$M/\approx$ cannot be collapsed further

Example

DFA for set of strings over $\{a, b\}$ containing at least three occurrences of three consecutive b's, overlapping permitted:



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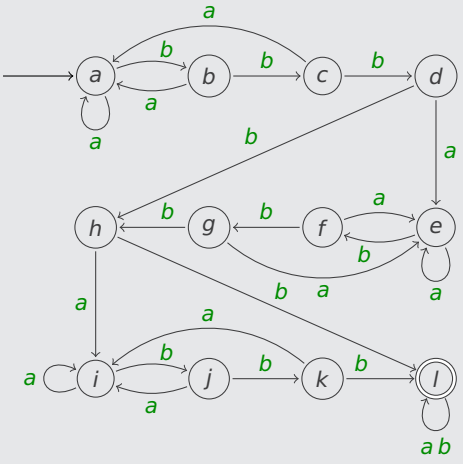

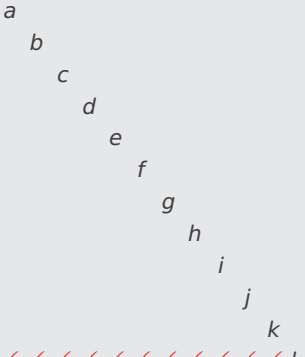

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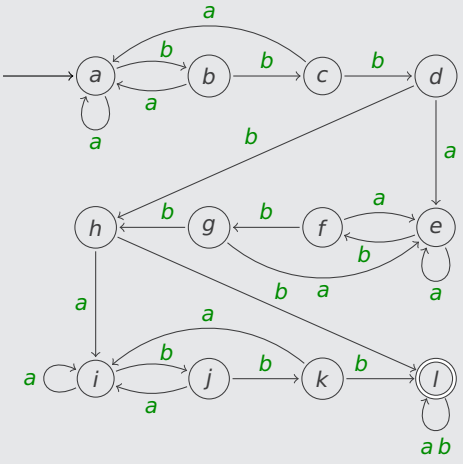



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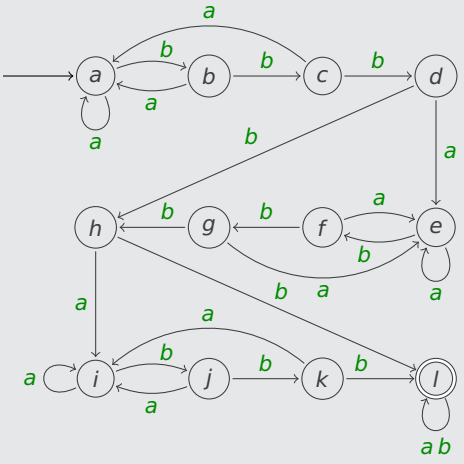

a											
	b										
		c									
✓	✓	✓	d								
			✓	e							
			✓		f						
✓	✓	✓		✓	✓	g					
✓	✓	✓	✓	✓	✓	✓	h				
			✓			✓	✓	i			
✓	✓	✓		✓	✓		✓	✓	j		
✓	✓	✓	✓	✓	✓	✓		✓	✓	k	
✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	l

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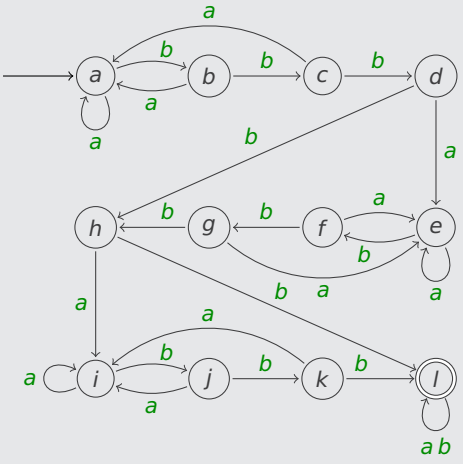

											a
										b	
✓	✓								c		
✓	✓	✓						d			
			✓	✓			e				
✓	✓			✓	✓	f					
✓	✓	✓			✓	✓	g				
✓	✓	✓	✓	✓	✓	✓	h				
✓	✓			✓	✓			✓	✓	i	
✓	✓	✓			✓	✓			✓	✓	j
✓	✓	✓	✓	✓	✓	✓				✓	✓ k
✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	l

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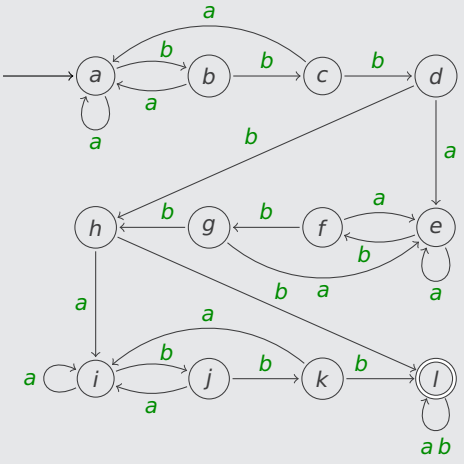

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✓	✓	✓	✓	✓	✓	✓	✓	i			
✓	✓	✓	✓	✓	✓	✓	✓	✓	j		
✓	✓	✓	✓	✓	✓	✓		✓	✓	k	
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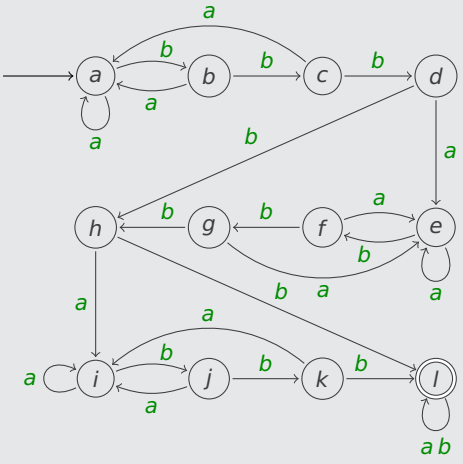

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✓ b
✓ ✓ c
✓ ✓ ✓ d
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✓ ✓ ✓ ✓ ✓ g
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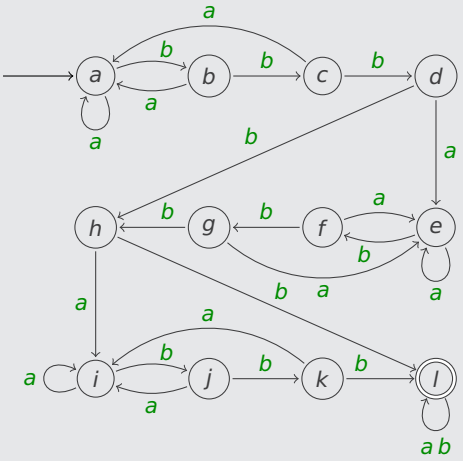

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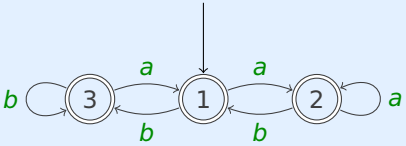
states d, g and h, k can be merged

Outline

1. Summary of Previous Lecture
2. Minimization
3. **Intermezzo**
4. Weak Monadic Second–Order Logic
5. Further Reading

Question

Which statements about the following DFA A are true ?



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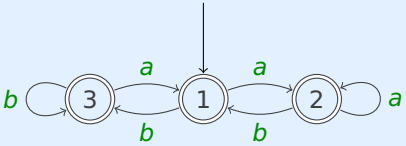

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$z = y + 1$ abbreviates $y < z \wedge \neg \exists x. (y < x \wedge x < z)$

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- $(\forall x. x = 0 \to X(x)) \wedge (\forall x. X(x) \to \exists y. x < y \wedge X(y))$ unsatisfiable

Definitions

- formula φ is satisfiable if $\alpha \vDash \varphi$ for some assignment α
- formula φ is valid if $\alpha \vDash \varphi$ for all assignments α
- model of formula φ is assignment α such that $\alpha \vDash \varphi$
- size of model α is smallest n such that
  1. $\alpha(x) < n$ for $x \in V_1$
  2. $\alpha(X) \subseteq \{0, \ldots, n-1\}$ for $X \in V_2$

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Definition

given alphabet Σ

- second–order variables $V_2 = \{P_a \mid a \in \Sigma\}$

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$\underline{x}$ for α_x

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$\Sigma = \{a, b\}$

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$\Sigma = \{a, b\}$

- $\underline{abba}\,(P_a) = \{0, 3\}$
- $\underline{abba}\,(P_b) = \{1, 2\}$

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Definitions

- given alphabet Σ and WMSO formula φ with free variables (exclusively) in $\{P_a \mid a \in \Sigma\}$

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- set $A \subseteq \Sigma^*$ is WMSO definable if $A = L(\varphi)$ for some WMSO formula φ

Examples

$\Sigma = \{a, b\}$

- regular set $L((a+b)^*ab(a+b)^*)$ is WMSO definable by formula

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Theorem

set $A \subseteq \Sigma^*$ is regular if and only if A is WMSO definable

Outline

1. Summary of Previous Lecture
2. Minimization
3. Intermezzo
4. Weak Monadic Second–Order Logic
5. **Further Reading**

Kozen

- Lectures 13 and 14

Kozen

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Important Concepts

- $\alpha \models \varphi$
- indistinguishable states
- minimization algorithm
- model
- MSO
- satisfiability
- validity
- weak monadic second–order logic (WMSO)
- WMSO definability

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homework for November 13