universität innsbruck

26W lecture 3

Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Outline

1. **Summary of Previous Lecture**
2. **Regular Expressions**
3. **Intermezzo**
4. **Homomorphisms**
5. **Decision Problems**
6. **Further Reading**

Definitions

- nondeterministic finite automaton (NFA) is quintuple $N = (Q, \Sigma, \Delta, S, F)$ with
  1. Q: finite set of states
  2. Σ: input alphabet
  3. $\Delta \colon Q \times \Sigma \to 2^Q$: transition function
  4. $S \subseteq Q$: set of start states
  5. $F \subseteq Q$: final (accept) states
- $\widehat{\Delta} \colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}(A, \epsilon) = A \qquad \widehat{\Delta}(A, xa) = \bigcup_{q \in \widehat{\Delta}(A,x)} \Delta(q, a)$$

- $x \in \Sigma^*$ is accepted by N if $\widehat{\Delta}(S, x) \cap F \neq \varnothing$

Theorem

every set accepted by NFA is regular

Definitions

- NFA with ϵ–transitions (NFA_ϵ) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that
  1. $\epsilon \notin \Sigma$
  2. $M_\epsilon = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\epsilon\}$
- ϵ–closure of set $A \subseteq Q$ is defined as $C_\epsilon(A) = \bigcup \{ \widehat{\Delta}_{N_\epsilon}(A, x) \mid x \in \{\epsilon\}^* \}$
- $\widehat{\Delta}_N \colon 2^Q \times \Sigma^* \to 2^Q$ is inductively defined by

$$\widehat{\Delta}_N(A, \epsilon) = C_\epsilon(A) \qquad \widehat{\Delta}_N(A, xa) = \bigcup \{ C_\epsilon(\Delta(q, a)) \mid q \in \widehat{\Delta}_N(A, x) \}$$

Theorem

- every set accepted by NFA_ϵ is regular
- regular sets are effectively closed under union, concatenation, and asterate

Automata

- (deterministic, nondeterministic, alternating) finite automata
- regular expressions
- (alternating) Büchi automata

Logic

- (weak) monadic second–order logic
- Presburger arithmetic
- linear–time temporal logic

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$$a \in \Sigma \qquad \epsilon \qquad \varnothing \qquad \beta + \gamma \qquad \beta\gamma \qquad \beta^*$$

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regular expressions α and β are equivalent $(\alpha \equiv \beta)$ if $L(\alpha) = L(\beta)$

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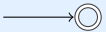
ϵ	$\{\epsilon\}$	
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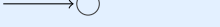
$\varnothing$	$\varnothing$	

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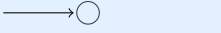
$\varnothing$	$\varnothing$	

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$L(\beta)$ and $L(\gamma)$ are regular according to induction hypothesis

$\Longrightarrow$ $L(\alpha)$ is regular according to closure properties of regular sets

Lemma (Arden's Lemma)

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Theorem

finite automata and regular expressions are equivalent:

$$\text{for all } A \subseteq \Sigma^* \quad A \text{ is regular} \quad \iff \quad A = L(\alpha) \text{ for some regular expression } \alpha$$

Proof ($\Longrightarrow$)

given NFA $N = (Q, \Sigma, \Delta, S, F)$ with $Q = \{1, \ldots, n\}$ and $S = \{1\}$

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- define system of equations

$$X_i = \left(\bigcup_{a \in \Sigma} \bigcup_{j \in \Delta(i,a)} \{a\} X_j \right) \cup \begin{cases} \{\epsilon\} & \text{if } i \in F \\ \varnothing & \text{otherwise} \end{cases}$$

with unknowns $X_1, \ldots, X_n$

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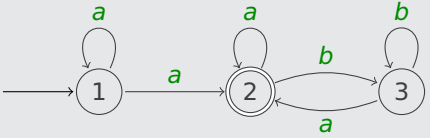
- transform X_1 into regular expression by successive substitution and Arden's lemma

Example



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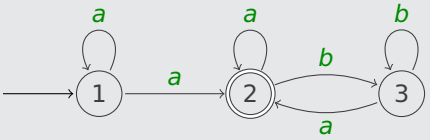

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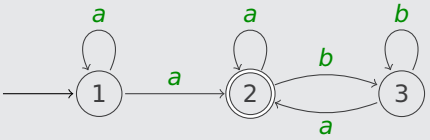

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$X_1 = a^*a\,X_2$ $\qquad$ $X_2 = a^*(b\,X_3 + \epsilon)$ $\qquad$ $X_3 = b^*a\,X_2$ $\qquad$ (Arden's lemma)

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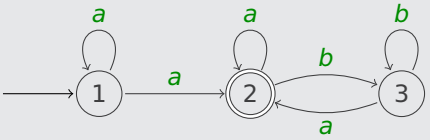

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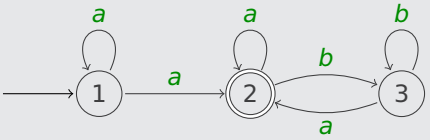
$$X_1 = a^*a\,X_2 \qquad X_2 = a^*(b\,X_3 + \epsilon) \qquad X_3 = b^*a\,X_2 \qquad \text{(Arden's lemma)}$$

$$X_1 = a^*a\,X_2 \qquad X_2 = a^*(bb^*a\,X_2 + \epsilon) \qquad \text{(substitute)}$$

Example



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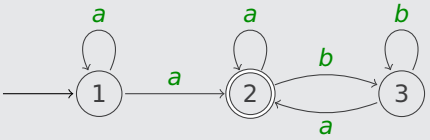

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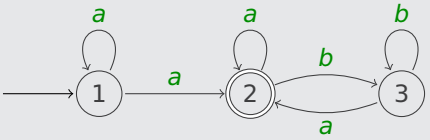

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Outline

1. Summary of Previous Lecture
2. Regular Expressions
3. **Intermezzo**
4. Homomorphisms
5. Decision Problems
6. Further Reading

Articify with session ID **3017 2066**

Question

Which of the following strings belong to $L((a + ab + aab)^*)$?

A ϵ

B all strings over $\{a, b\}$ that start with a

C $ababa$

D all strings over $\{a, b\}$ that start with a and do not contain two consecutive b's

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regular sets are effectively closed under homomorphic image and preimage

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- homomorphism is mapping $h\colon \Sigma^* \to \Gamma^*$ such that

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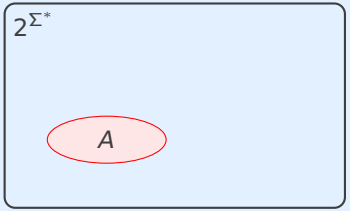
- homomorphism $h\colon \Sigma^* \to \Gamma^*$



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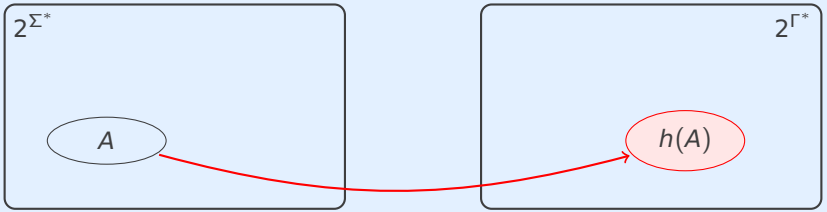

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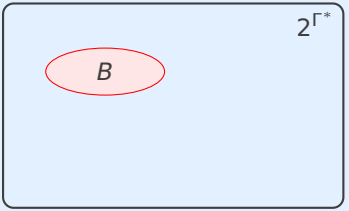

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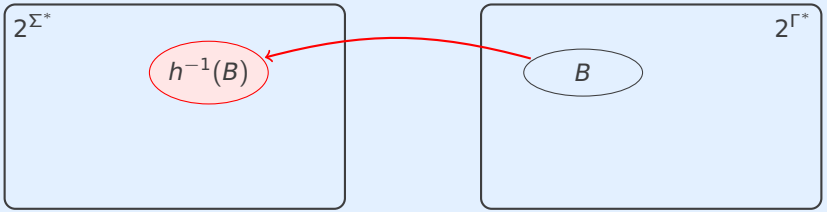

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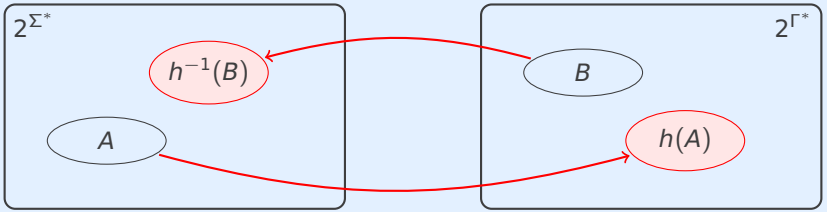

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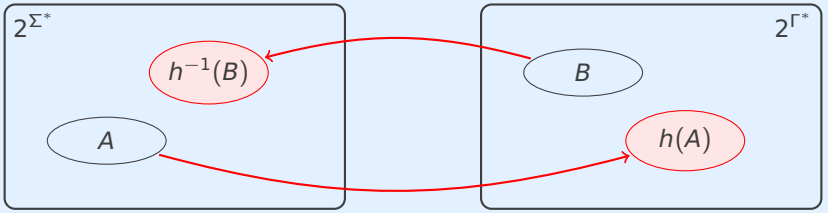

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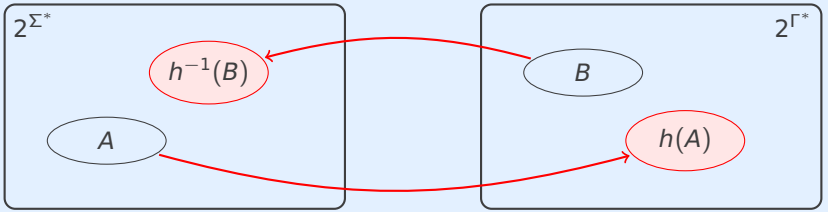
$\Sigma = \Gamma = \{0,1\} \quad h(0) = 11 \quad h(1) = 1$



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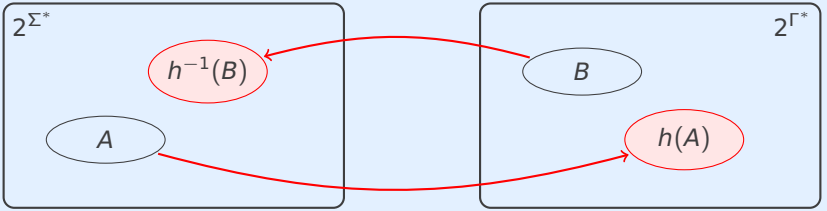

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Proof

- NFA $M = (Q, \Gamma, \Delta, S, F)$
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- NFA $M = (Q, \Gamma, \Delta, S, F)$
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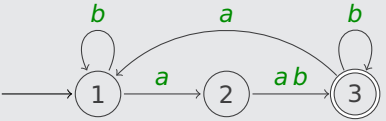
Proof

- NFA $M = (Q, \Gamma, \Delta, S, F)$
- homomorphism $h \colon \Sigma^* \to \Gamma^*$
- $h^{-1}(L(M)) = L(M')$ for NFA $M' = (Q, \Sigma, \Delta', S, F)$ with $\Delta'(q, a) = \widehat{\Delta}(\{q\}, h(a))$
- claim: $\widehat{\Delta'}(A, x) = \widehat{\Delta}(A, h(x))$ for all $A \subseteq Q$ and $x \in \Sigma^*$

  proof of claim: easy induction on $|x|$

Example

- DFA M



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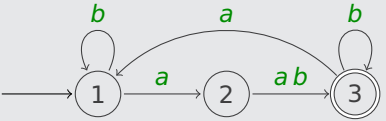

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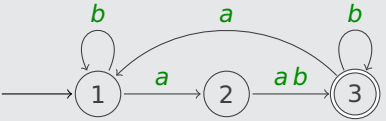

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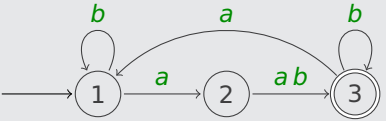

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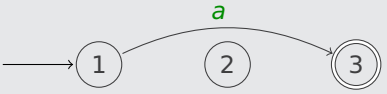

$\delta'(1,a) = \widehat{\delta}(1,aa) = 3$

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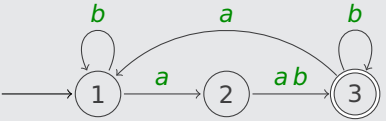

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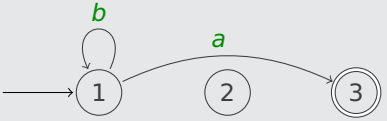

$\delta'(1,b) = \widehat{\delta}(1,\epsilon) = 1$

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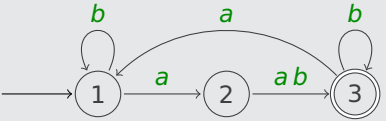

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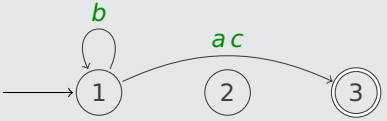

$\delta'(1,c) = \widehat{\delta}(1,bab) = 3$

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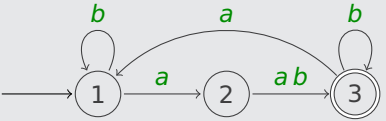

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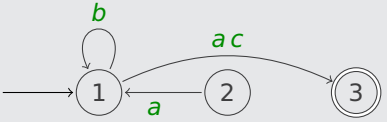

$\delta'(2,a) = \widehat{\delta}(2,aa) = 1$

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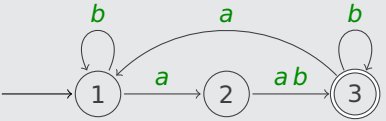

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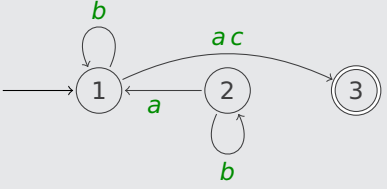

$\delta'(2,b) = \widehat{\delta}(2,\epsilon) = 2$

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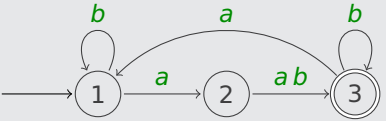

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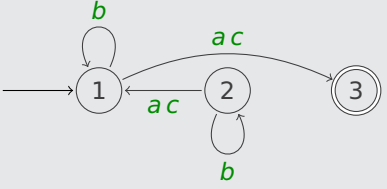

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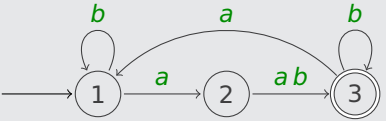

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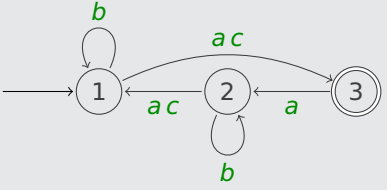

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Example

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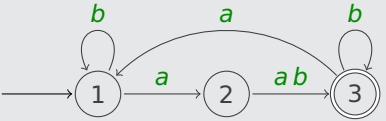
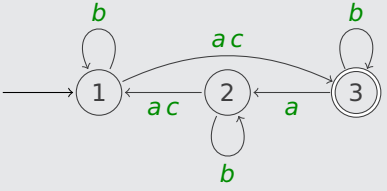


- homomorphism $h\colon \{a,b,c\}^* \to \{a,b\}^*$

$$h(a) = aa \qquad h(b) = \epsilon \qquad h(c) = bab$$

- DFA M'



$\delta'(3,b) = \widehat{\delta}(3,\epsilon) = 3$

Example

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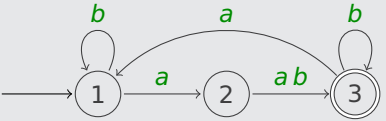

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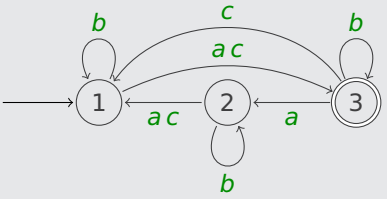

$\delta'(3,c) = \widehat{\delta}(3,bab) = 1$

Theorem

regular sets are effectively closed under homomorphic image and preimage

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Proof

- regular expression α over Σ
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- regular expression α over Σ
- homomorphism $h\colon \Sigma^* \to \Gamma^*$
- $h(L(\alpha)) = L(\alpha')$ for regular expression α' defined inductively:

$$a' = h(a) \quad \text{for } a \in \Sigma \qquad (\beta + \gamma)' = \beta' + \gamma'$$
$$\epsilon' = \epsilon \qquad (\beta\gamma)' = \beta'\gamma'$$
$$\varnothing' = \varnothing \qquad \beta^{*\prime} = \beta'^{*}$$

Definitions

- Hamming distance $H(x, y)$ is number of places where bit strings x and y differ
- if $|x| \neq |y|$ then $H(x, y) = \infty$
- $N_k(A) = \{x \in \{0,1\}^* \mid H(x, y) \leqslant k \text{ for some } y \in A\}$

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0	0	1	1

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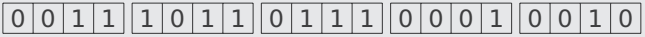

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1	0	0	1	1	0	1	0	0	1	0	1	0	1	1	0	0	0	0	0				

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0	0	0	0	0	0	0	1	0	0	1	0	0	0	1	1	0	1	0	0	0	1	0	1
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0	0	0	0	0	0	0	1	0	0	1	0	0	0	1	1				0	1	0	1	
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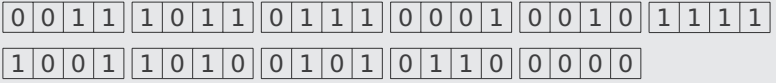

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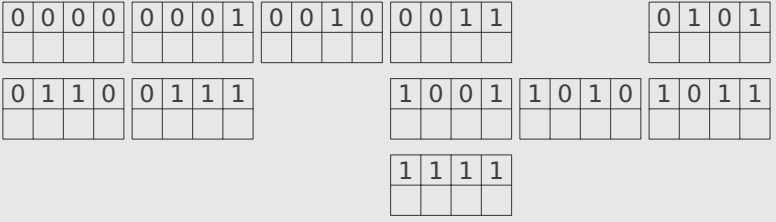

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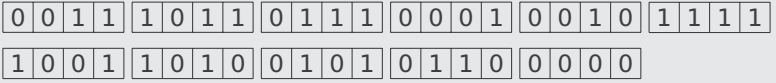

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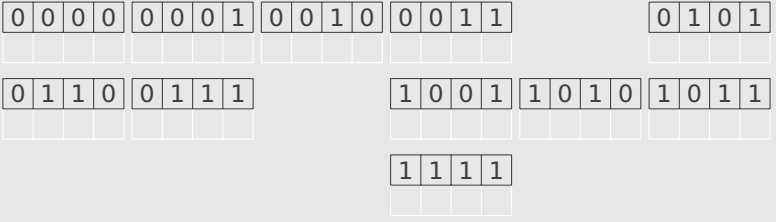

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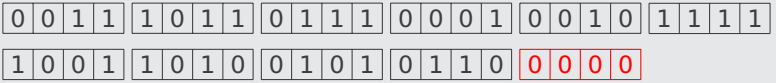

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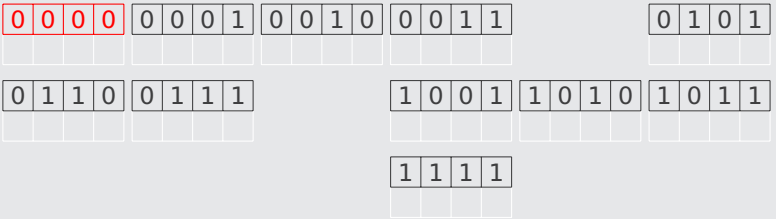

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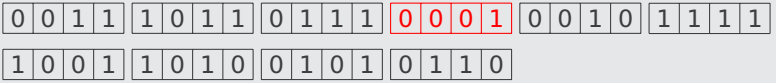

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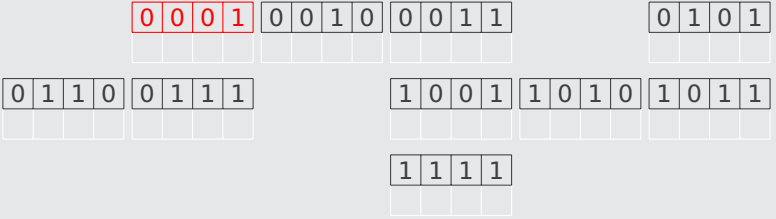

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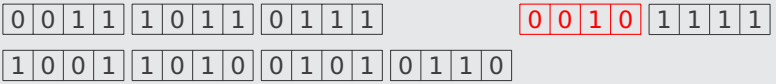

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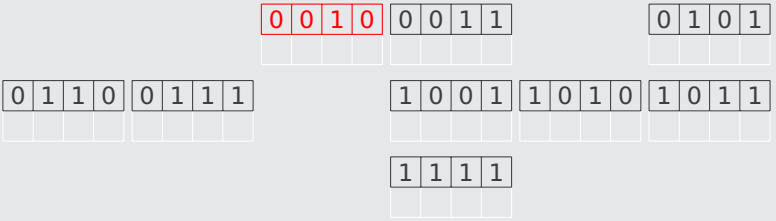

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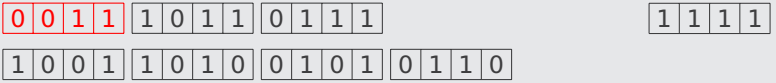

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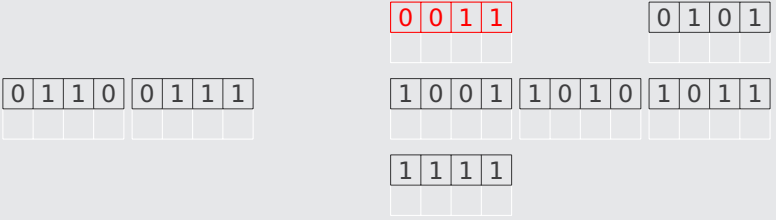

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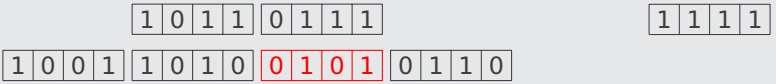

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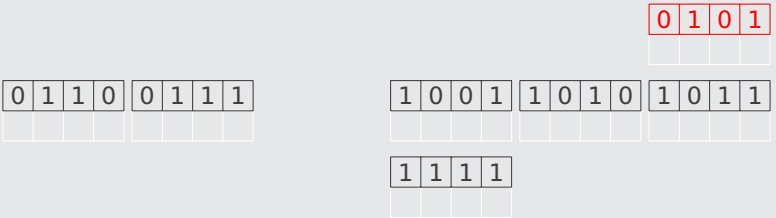

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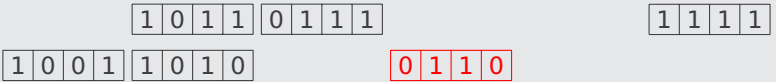

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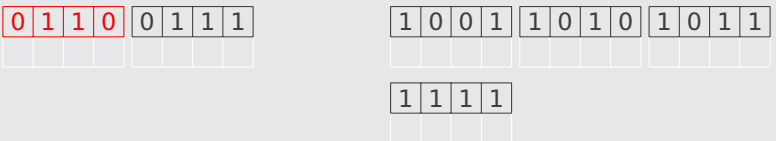

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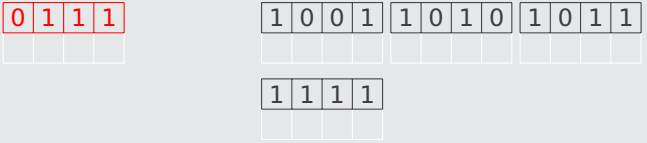

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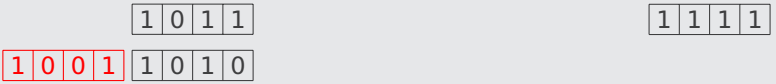

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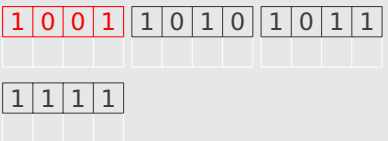

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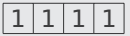

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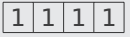

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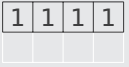

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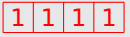

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Outline

1. Summary of Previous Lecture
2. Regular Expressions
3. Intermezzo
4. Homomorphisms
5. **Decision Problems**
6. Further Reading

Remark

most decision problems concerning regular sets are decidable

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Theorem

problems

instance: DFA M and string x

question: $x \in L(M)$?

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representation of regular sets (DFA, NFA, regular expression) may affect complexity of decision problems

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Kozen

- Lecture 7 – 10

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Important Concepts

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- homomorphism
- homomorphic image
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homework for October 30 & November 6