



Automata and Logic

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Outline

- 1. Summary of Previous Lecture**
- 2. Nondeterministic Finite Automata**
- 3. Epsilon Transitions**
- 4. Intermezzo**
- 5. Closure Properties**
- 6. Hamming Distance**
- 7. Further Reading**

Definitions

► **deterministic finite automaton (DFA)** is quintuple $M = (Q, \Sigma, \delta, s, F)$ with

- ① Q : finite set of **states**
- ② Σ : **input alphabet**
- ③ $\delta: Q \times \Sigma \rightarrow Q$: **transition function**
- ④ $s \in Q$: **start state**
- ⑤ $F \subseteq Q$: **final (accept) states**

► $\hat{\delta}: Q \times \Sigma^* \rightarrow Q$ is inductively defined by

$$\hat{\delta}(q, \epsilon) = q$$

$$\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is **accepted** by M if $\hat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is **rejected** by M if $\hat{\delta}(s, x) \notin F$
- language accepted by M : $L(M) = \{x \in \Sigma^* \mid \hat{\delta}(s, x) \in F\}$

Definition

set $A \subseteq \Sigma^*$ is **regular** if $A = L(M)$ for some DFA M

Theorem

regular sets are **effectively** closed under **intersection**, **union**, and **complement**

Automata

- ▶ (deterministic, **nondeterministic**, alternating) **finite automata**
- ▶ regular expressions
- ▶ (alternating) Büchi automata

Logic

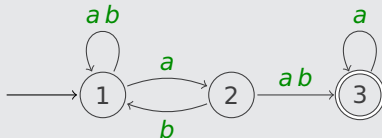
- ▶ (weak) monadic second-order logic
- ▶ Presburger arithmetic
- ▶ linear-time temporal logic

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Example

NFA $M = (Q, \Sigma, \Delta, S, F)$



1 $Q = \{1, 2, 3\}$

2 $\Sigma = \{a, b\}$

3 $\Delta: Q \times \Sigma \rightarrow 2^Q$

4 $S = \{1\}$

5 $F = \{3\}$

Δ	a	b
1	$\{1, 2\}$	$\{1\}$
2	$\{3\}$	$\{1, 3\}$
3	$\{3\}$	$\emptyset$

Definitions

► **nondeterministic finite automaton (NFA)** is quintuple $N = (Q, \Sigma, \Delta, S, F)$ with

- ① Q : finite set of states
- ② Σ : input alphabet
- ③ $\Delta: Q \times \Sigma \rightarrow 2^Q$: transition function
- ④ $S \subseteq Q$: set of start states
- ⑤ $F \subseteq Q$: final (accept) states

► $\hat{\Delta}: 2^Q \times \Sigma^* \rightarrow 2^Q$ is inductively defined by

$$\hat{\Delta}(A, \epsilon) = A$$

$$\hat{\Delta}(A, xa) = \bigcup_{q \in \hat{\Delta}(A, x)} \Delta(q, a)$$

► $x \in \Sigma^*$ is accepted by N if $\hat{\Delta}(S, x) \cap F \neq \emptyset$

► $L(N) = \{x \in \Sigma^* \mid \hat{\Delta}(S, x) \cap F \neq \emptyset\}$

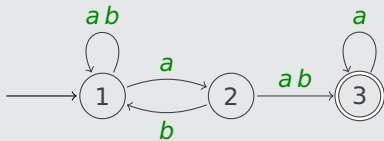
Theorem

every set accepted by NFA is regular

Proof (subset construction)

- ▶ NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- ▶ $L(N) = L(M)$ for DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ with
 - ① $Q_M = 2^{Q_N}$
 - ② $\delta_M(A, a) = \widehat{\Delta}_N(A, a)$ for all $A \subseteq Q_N$ and $a \in \Sigma$
 - ③ $s_M = S_N$
 - ④ $F_M = \{A \subseteq Q_N \mid A \cap F_N \neq \emptyset\}$
- ▶ claim: $\widehat{\delta}_M(A, x) = \widehat{\Delta}_N(A, x)$ for all $A \subseteq Q_N$ and $x \in \Sigma^*$
proof of claim: easy induction on $|x|$

Example



$A = \emptyset$ $E = \{1, 2\}$

$B = \{1\}$ $F = \{1, 3\}$

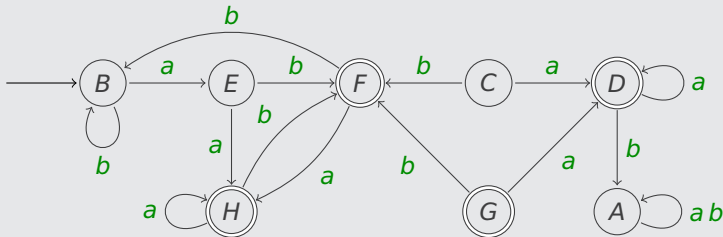
$C = \{2\}$ $G = \{2, 3\}$

$D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba

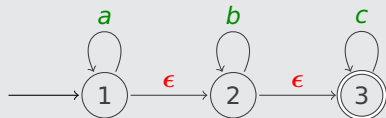
remove inaccessible states



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Example



$$C_{\epsilon}(\{1\}) = \{1, 2, 3\}$$

$$\hat{\Delta}(\{1\}, b) = \{2, 3\}$$

Definitions

► **NFA with ϵ -transitions** (NFA_{ϵ}) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that

① $\epsilon \notin \Sigma$

② $N_{\epsilon} = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ is NFA over alphabet $\Sigma \cup \{\epsilon\}$

► **ϵ -closure** of set $A \subseteq Q$ is defined as $C_{\epsilon}(A) = \bigcup \{ \hat{\Delta}_{N_{\epsilon}}(A, x) \mid x \in \{\epsilon\}^* \}$

► $\hat{\Delta}_N: 2^Q \times \Sigma^* \rightarrow 2^Q$ is inductively defined by

$$\hat{\Delta}_N(A, \epsilon) = C_{\epsilon}(A)$$

$$\hat{\Delta}_N(A, xa) = \bigcup \{ C_{\epsilon}(\Delta(q, a)) \mid q \in \hat{\Delta}_N(A, x) \}$$

Lemma

$C_\epsilon(A)$ is least extension of A that is closed under ϵ -transitions:

$$q \in C_\epsilon(A) \implies \Delta_{N_\epsilon}(q, \epsilon) \subseteq C_\epsilon(A)$$

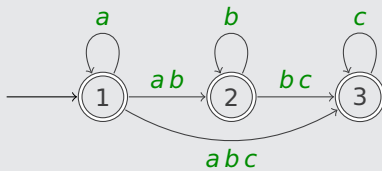
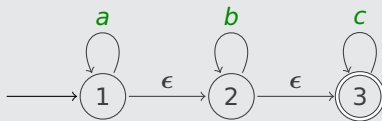
Theorem

every set accepted by NFA_ϵ is regular

Proof (construction)

- ▶ $NFA_\epsilon N_1 = (Q, \Sigma, \epsilon, \Delta_1, S, F_1)$
- ▶ $L(N_1) = L(N_2)$ for $NFA N_2 = (Q, \Sigma, \Delta_2, S, F_2)$ with
 - ① $\Delta_2(q, a) = \hat{\Delta}_1(\{q\}, a)$ for all $q \in Q$ and $a \in \Sigma$
 - ② $F_2 = \{q \mid C_\epsilon(\{q\}) \cap F_1 \neq \emptyset\}$

Example

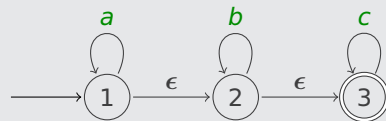


Example

NFA_ε $N_1 = (\{1, 2, 3\}, \{a, b, c\}, \epsilon, \Delta_1, \{1\}, \{3\})$ with

► Δ_1

	a	b	c	ϵ
1	$\{1\}$	$\emptyset$	$\emptyset$	$\{2\}$
2	$\emptyset$	$\{2\}$	$\emptyset$	$\{3\}$
3	$\emptyset$	$\emptyset$	$\{3\}$	$\emptyset$

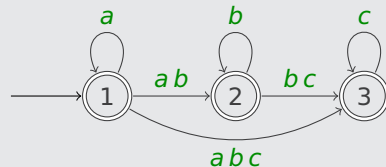


NFA $N_2 = (\{1, 2, 3\}, \{a, b, c\}, \Delta_2, \{1\}, F_2)$ with

► $F_2 = \{q \mid C_\epsilon(\{q\}) \cap \{3\} \neq \emptyset\} = \{q \mid 3 \in C_\epsilon(\{q\})\} = \{1, 2, 3\}$

► Δ_2

	a	b	c
1	$\{1, 2, 3\}$	$\{2, 3\}$	$\{3\}$
2	$\emptyset$	$\{2, 3\}$	$\{3\}$
3	$\emptyset$	$\emptyset$	$\{3\}$



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Question

What is the language accepted by the NFA_{ϵ} given by the following transition table ?

	ϵ	a	b
$\rightarrow$ 1	$\emptyset$	$\{2\}$	$\emptyset$
2	$\{3\}$	$\emptyset$	$\emptyset$
3	$\emptyset$	$\{4\}$	$\{2\}$
4 F	$\{1\}$	$\emptyset$	$\emptyset$

- A** the set of all strings over $\{a, b\}$ starting and ending with a
- B** $\{xyx \mid x \in \{a\} \text{ and } y \in \{b\}^*\}$
- ☒ **C** $\{\{a\}\{b\}^*\{a\}\}^+$
- D** $\{xyz \mid x, z \in \{a, b\} \text{ and } y \in \{aab\}^*\}$



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Theorem

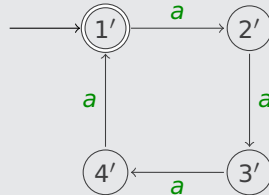
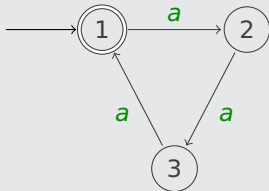
regular sets are **effectively** closed under **union**, concatenation, and asterate

Proof

- ▶ $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
 $B = L(N_2)$ for NFA $N_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$
- ▶ without loss of generality $Q_1 \cap Q_2 = \emptyset$
- ▶ **$A \cup B = L(N)$** for NFA $N = (Q, \Sigma, \Delta, S, F)$ with
 - ① $Q = Q_1 \cup Q_2$
 - ② $S = S_1 \cup S_2$
 - ③ $F = F_1 \cup F_2$
 - ④ $\Delta(q, a) = \begin{cases} \Delta_1(q, a) & \text{if } q \in Q_1 \\ \Delta_2(q, a) & \text{if } q \in Q_2 \end{cases}$

Example

$$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\} \cup \{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$$



$$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3 \text{ or } 4\}$$

Theorem

regular sets are **effectively** closed under union, **concatenation**, and asterate

Proof

- ▶ $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
 $B = L(N_2)$ for NFA $N_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$
- ▶ without loss of generality $Q_1 \cap Q_2 = \emptyset$
- ▶ **AB** = $L(N)$ for NFA _{ϵ} $N = (Q, \Sigma, \epsilon, \Delta, S_1, F_2)$ with

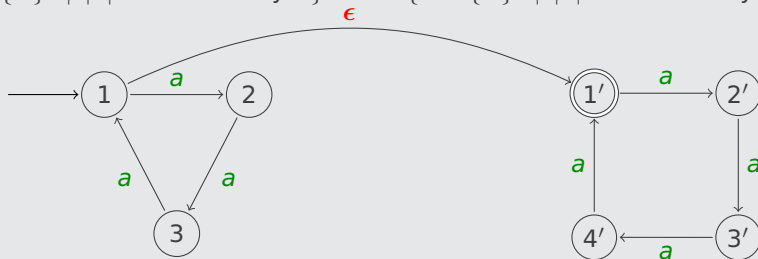
$$\textcircled{1} \quad Q = Q_1 \cup Q_2$$

$$\textcircled{2} \quad \Delta(q, a) = \begin{cases} \Delta_1(q, a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma \\ \Delta_2(q, a) & \text{if } q \in Q_2 \text{ and } a \in \Sigma \\ S_2 & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \emptyset & \text{otherwise} \end{cases}$$

Example

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



$\{x \in \{a\}^* \mid |x| \notin \{1, 2, 5\}\}$

Theorem

regular sets are **effectively** closed under union, concatenation, and **asterate**

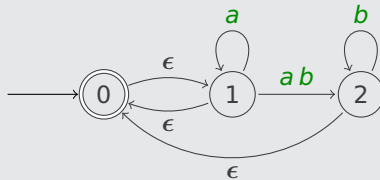
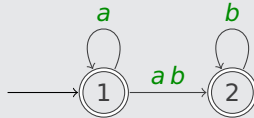
Proof

- ▶ $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$
- ▶ $A^* = L(N)$ for NFA _{ϵ} $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ with
 - ① $Q = Q_1 \uplus \{s\}$
 - ② $S = \{s\}$
 - ③ $F = \{s\}$
 - ④ $\Delta(q, a) = \begin{cases} \Delta_1(q, a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma \\ S_1 & \text{if } q = s \text{ and } a = \epsilon \\ S & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \emptyset & \text{otherwise} \end{cases}$

Example

$$\{a\}^* \{b\}^*$$

$$(\{a\}^* \{b\}^*)^* = \{a, b\}^*$$



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Definitions

- ▶ **Hamming distance** $H(x, y)$ is number of places where bit strings x and y differ
- ▶ if $|x| \neq |y|$ then $H(x, y) = \infty$
- ▶ $N_k(A) = \{x \in \{0, 1\}^* \mid H(x, y) \leq k \text{ for some } y \in A\}$

Lemma

$A \subseteq \{0, 1\}^*$ is regular $\implies N_2(A)$ is regular

Lemma

$A \subseteq \{0, 1\}^*$ is regular $\implies N_2(A)$ is regular

Proof

- ▶ $A = L(M)$ for DFA $M = (Q_M, \{0, 1\}, \delta_M, s_M, F_M)$
- ▶ define NFA $N = (Q_N, \{0, 1\}, \Delta_N, S_N, F_N)$ with
 - ① $Q_N = Q_M \times \{0, 1, 2\}$
 - ② $\Delta_N((p, 0), a) = \{(q, 0) \mid \delta_M(p, a) = q\} \cup \{(q, 1) \mid \delta_M(p, b) = q \text{ for some } b \neq a\}$
 $\Delta_N((p, 1), a) = \{(q, 1) \mid \delta_M(p, a) = q\} \cup \{(q, 2) \mid \delta_M(p, b) = q \text{ for some } b \neq a\}$
 $\Delta_N((p, 2), a) = \{(q, 2) \mid \delta_M(p, a) = q\}$ for all $a \in \Sigma$
 - ③ $S_N = \{(s_M, 0)\}$
 - ④ $F_N = F_M \times \{0, 1, 2\}$

- ▶ key property:

$$(q, j) \in \hat{\Delta}_N(\{(p, i)\}, y) \iff \widehat{\delta}_M(p, x) = q \text{ for some } x \in \{0, 1\}^*$$

for all $p, q \in Q_M$, $y \in \{0, 1\}^*$, $i, j \in \{0, 1, 2\}$ such that $|x| = |y|$ and $H(x, y) = j - i$

- ▶ $N_2(A) = \{y \mid H(y, x) \leq 2 \text{ for some } x \in A\}$
 $= \{y \mid H(y, x) = k \text{ for some } x \in A \text{ and } k \in \{0, 1, 2\}\}$
 $= \{y \mid H(y, x) = k \text{ and } \widehat{\delta}_M(s_M, x) = q \text{ for some } x \in A, k \in \{0, 1, 2\} \text{ and } q \in F_M\}$
 $= \{y \mid (q, k) \in \hat{\Delta}_N(\{(s_M, 0)\}, y) \text{ for some } q \in F_M \text{ and } k \in \{0, 1, 2\}\}$
 $= \{y \mid (q, k) \in \hat{\Delta}_N(\{(s_M, 0)\}, y) \text{ for some } (q, k) \in F_N\}$
 $= \{y \mid \hat{\Delta}_N(\{(s_M, 0)\}, y) \cap F_N \neq \emptyset\}$
 $= L(N)$

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► Lecture 5 and 6

Important Concepts

- ϵ -transition
- ϵ -closure
- asterate
- Hamming distance
- NFA
- NFA_ϵ
- subset construction

homework for October 23