



Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Outline

- 1. Summary of Previous Lecture**
- 2. Nondeterministic Finite Automata**
- 3. Epsilon Transitions**
- 4. Intermezzo**
- 5. Closure Properties**
- 6. Hamming Distance**
- 7. Further Reading**

Definitions

► **deterministic finite automaton (DFA)** is quintuple $M = (Q, \Sigma, \delta, s, F)$ with

- ① Q : finite set of **states**
- ② Σ : **input alphabet**
- ③ $\delta: Q \times \Sigma \rightarrow Q$: **transition function**
- ④ $s \in Q$: **start state**
- ⑤ $F \subseteq Q$: **final (accept) states**

► $\hat{\delta}: Q \times \Sigma^* \rightarrow Q$ is inductively defined by

$$\hat{\delta}(q, \epsilon) = q$$

$$\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is **accepted** by M if $\hat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is **rejected** by M if $\hat{\delta}(s, x) \notin F$
- language accepted by M : $L(M) = \{x \in \Sigma^* \mid \hat{\delta}(s, x) \in F\}$

Definition

set $A \subseteq \Sigma^*$ is **regular** if $A = L(M)$ for some DFA M

Theorem

regular sets are **effectively** closed under **intersection**, **union**, and **complement**

Automata

- ▶ (deterministic, **nondeterministic**, alternating) **finite automata**
- ▶ regular expressions
- ▶ (alternating) Büchi automata

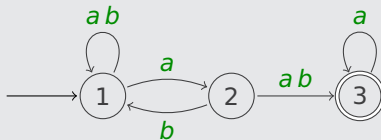
Logic

- ▶ (weak) monadic second-order logic
- ▶ Presburger arithmetic
- ▶ linear-time temporal logic

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Example



► **nondeterministic finite automaton (NFA)** is quintuple $N = (Q, \Sigma, \Delta, S, F)$ with

① Q : finite set of states

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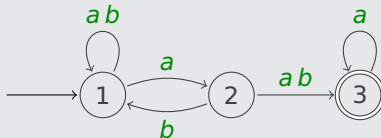
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Example

NFA $M = (Q, \Sigma, \Delta, S, F)$



1 $Q = \{1, 2, 3\}$

2 $\Sigma = \{a, b\}$

3 $\Delta: Q \times \Sigma \rightarrow 2^Q$

4 $S = \{1\}$

5 $F = \{3\}$

Δ	a	b
1	$\{1, 2\}$	$\{1\}$
2	$\{3\}$	$\{1, 3\}$
3	$\{3\}$	$\emptyset$

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$$\hat{\Delta}(A, \epsilon) = A$$

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► $L(N) = \{x \in \Sigma^* \mid \hat{\Delta}(S, x) \cap F \neq \emptyset\}$

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every set accepted by NFA is regular

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- ▶ NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- ▶ $L(N) = L(M)$ for DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ with

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 - ② $\delta_M(A, a) = \hat{\Delta}_N(A, a)$ for all $A \subseteq Q_N$ and $a \in \Sigma$

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- ▶ claim: $\widehat{\delta}_M(A, x) = \widehat{\Delta}_N(A, x)$ for all $A \subseteq Q_N$ and $x \in \Sigma^*$
proof of claim: easy induction on $|x|$

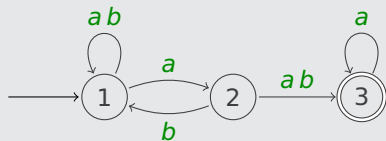
Theorem

every set accepted by NFA is regular

Proof (subset construction)

- ▶ NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$
- ▶ $L(N) = L(M)$ for DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ with
 - ① $Q_M = 2^{Q_N}$
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Example



$$A = \emptyset \quad E = \{1, 2\}$$

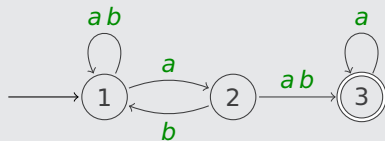
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$$C = \{2\} \quad G = \{2, 3\}$$

$$D = \{3\} \quad H = \{1, 2, 3\}$$

	<i>a</i>	<i>b</i>		<i>a</i>	<i>b</i>
<i>A</i>			<i>E</i>		
<i>B</i>			<i>F</i>		
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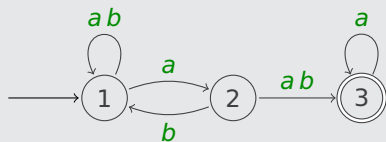
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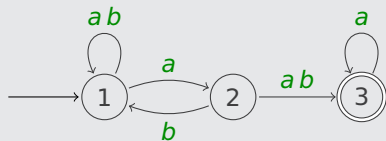
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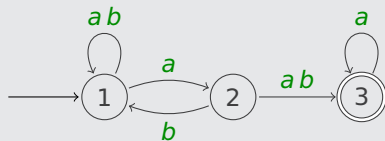
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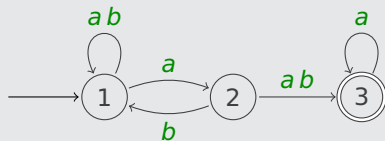
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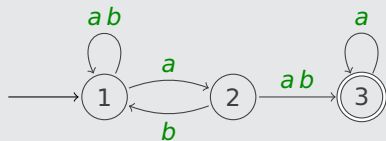
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<i>B</i>	<i>E</i>	<i>B</i>	<i>F</i>		
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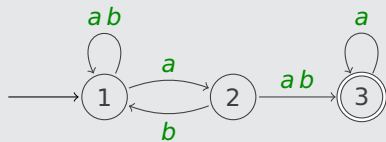
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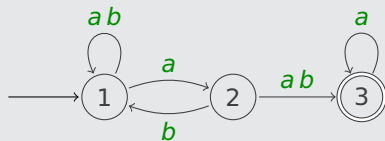
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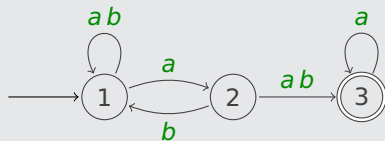
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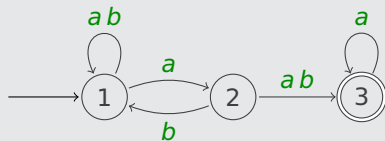
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<i>B</i>	<i>E</i>	<i>B</i>	<i>F</i>	<i>H</i>	<i>B</i>
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<i>D</i>	<i>D</i>	<i>A</i>	<i>H</i>	<i>H</i>	<i>F</i>

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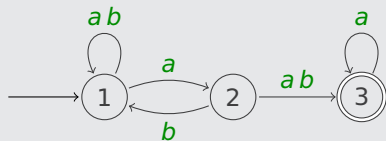
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<i>B</i>	<i>E</i>	<i>B</i>	<i>F</i>	<i>H</i>	<i>B</i>
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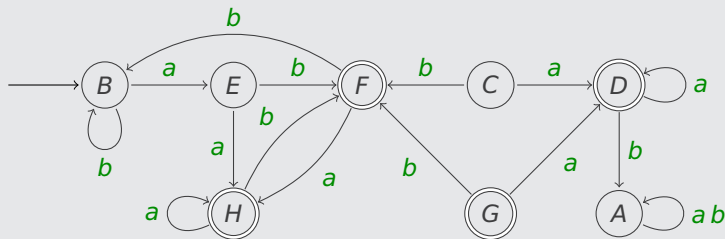
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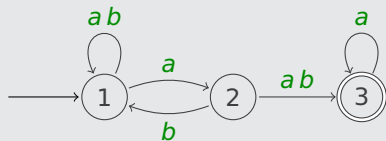
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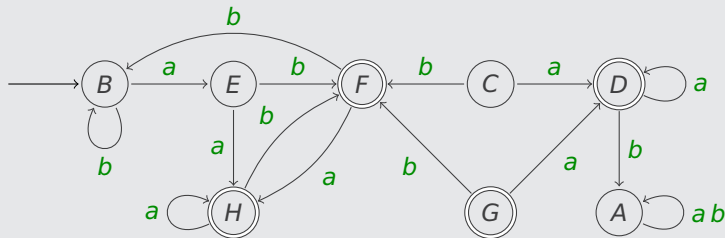
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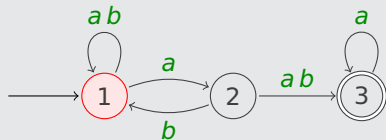
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	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



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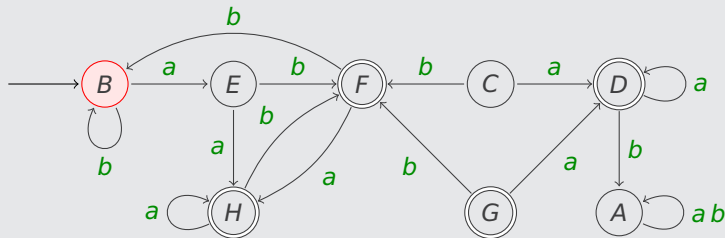
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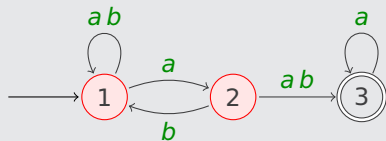
$D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



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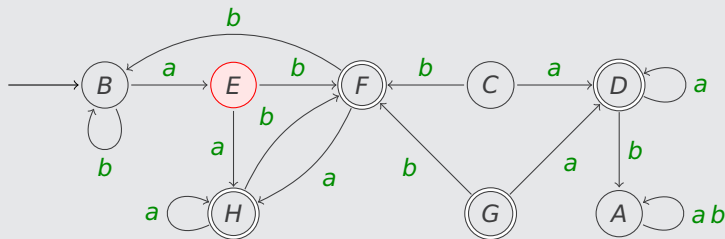
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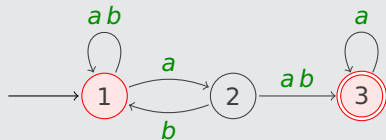
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	a	b		a	b
A	A	A	E	H	F
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C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



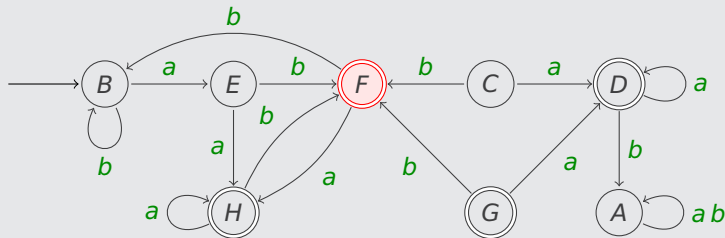
Example



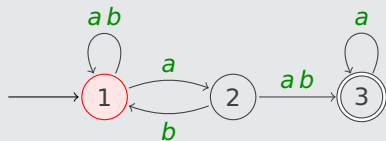
$A = \emptyset$ $E = \{1, 2\}$
 $B = \{1\}$ $F = \{1, 3\}$
 $C = \{2\}$ $G = \{2, 3\}$
 $D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbbaababbabbbaababba



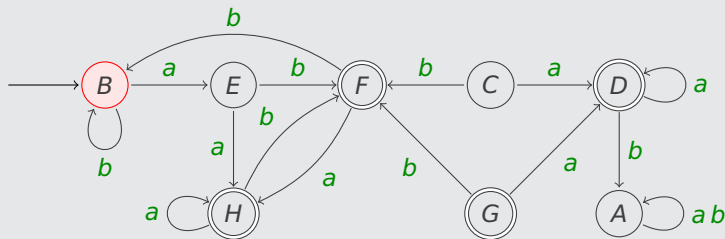
Example



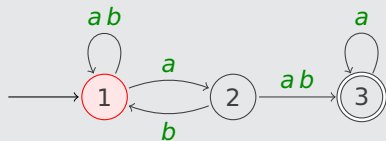
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 $C = \{2\}$ $G = \{2, 3\}$
 $D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



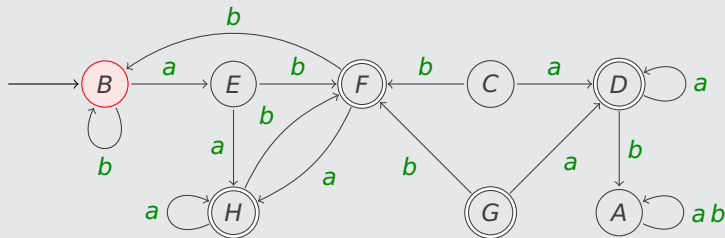
Example



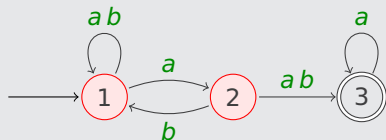
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 $B = \{1\}$ $F = \{1, 3\}$
 $C = \{2\}$ $G = \{2, 3\}$
 $D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbb aababbabbbaababba



Example



$A = \emptyset$ $E = \{1, 2\}$

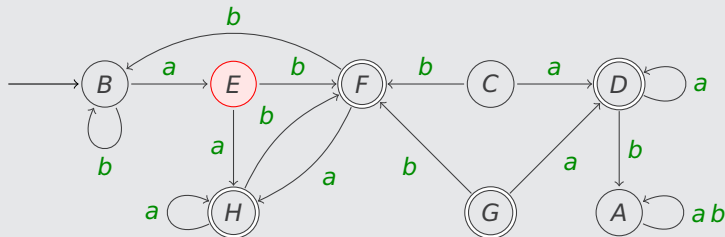
$B = \{1\}$ $F = \{1, 3\}$

$C = \{2\}$ $G = \{2, 3\}$

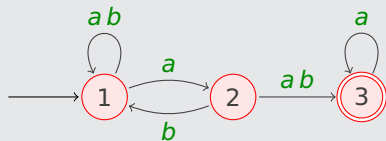
$D = \{3\}$ $H = \{1, 2, 3\}$

	<i>a</i>	<i>b</i>		<i>a</i>	<i>b</i>
<i>A</i>	<i>A</i>	<i>A</i>	<i>E</i>	<i>H</i>	<i>F</i>
<i>B</i>	<i>E</i>	<i>B</i>	<i>F</i>	<i>H</i>	<i>B</i>
<i>C</i>	<i>D</i>	<i>F</i>	<i>G</i>	<i>D</i>	<i>F</i>
<i>D</i>	<i>D</i>	<i>A</i>	<i>H</i>	<i>H</i>	<i>F</i>

abbbaababbabbbaababba



Example



$A = \emptyset$ $E = \{1, 2\}$

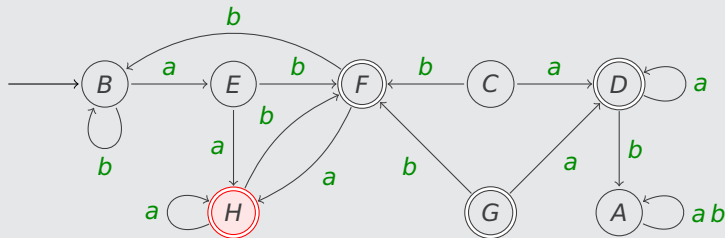
$B = \{1\}$ $F = \{1, 3\}$

$C = \{2\}$ $G = \{2, 3\}$

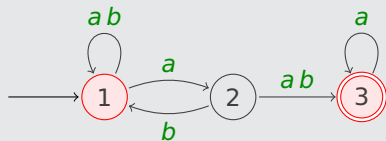
$D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



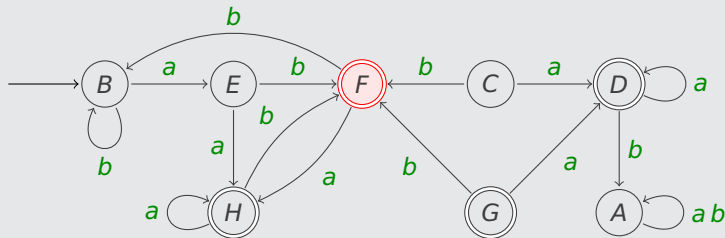
Example



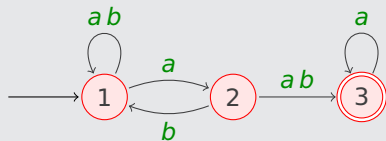
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 $C = \{2\}$ $G = \{2, 3\}$
 $D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



Example



$A = \emptyset$ $E = \{1, 2\}$

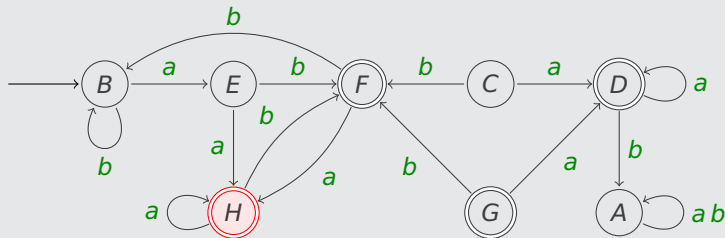
$B = \{1\}$ $F = \{1, 3\}$

$C = \{2\}$ $G = \{2, 3\}$

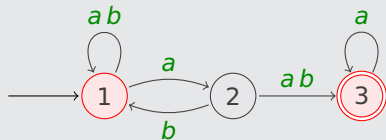
$D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbbaabaabbabbbaababba



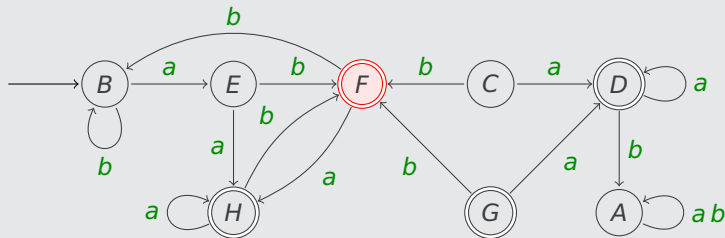
Example



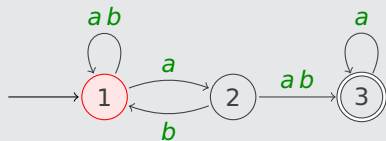
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 $C = \{2\}$ $G = \{2, 3\}$
 $D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



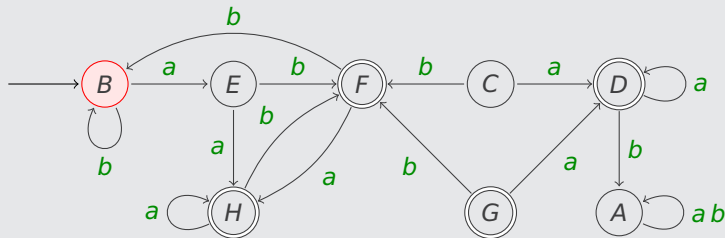
Example



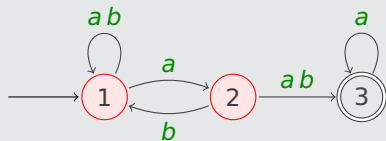
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	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



Example



$A = \emptyset$ $E = \{1, 2\}$

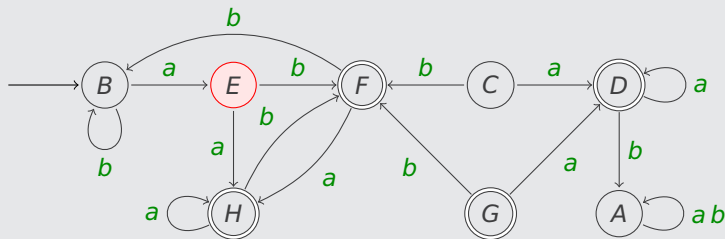
$B = \{1\}$ $F = \{1, 3\}$

$C = \{2\}$ $G = \{2, 3\}$

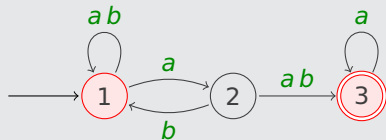
$D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



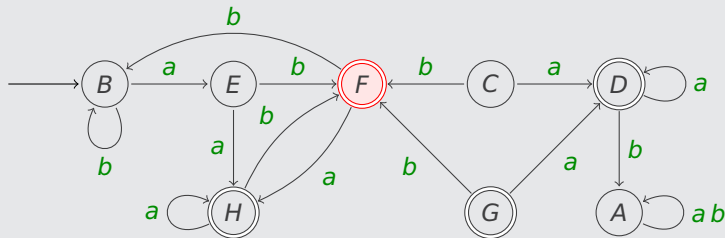
Example



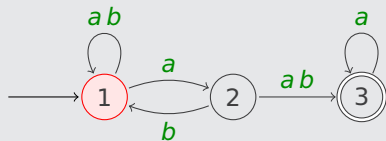
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	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbaababba



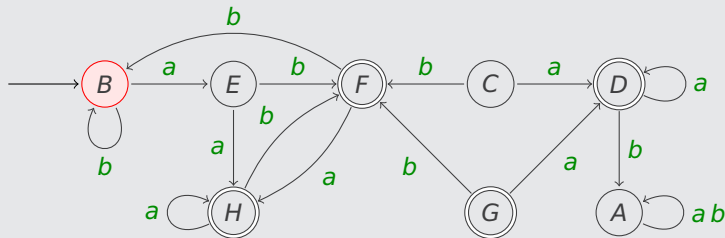
Example



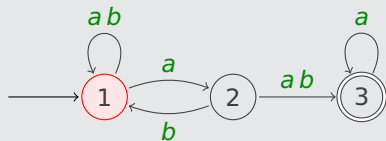
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 $C = \{2\}$ $G = \{2, 3\}$
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	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



Example



$A = \emptyset$ $E = \{1, 2\}$

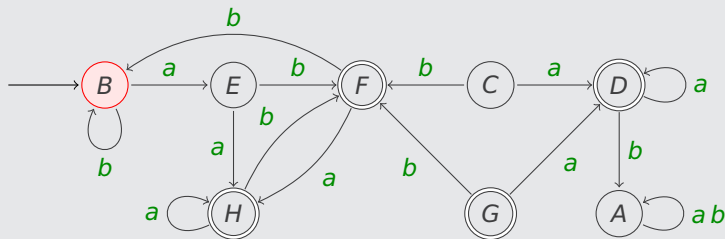
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$C = \{2\}$ $G = \{2, 3\}$

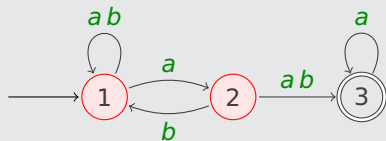
$D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbbaababbabbbbaababba



Example



$A = \emptyset$ $E = \{1, 2\}$

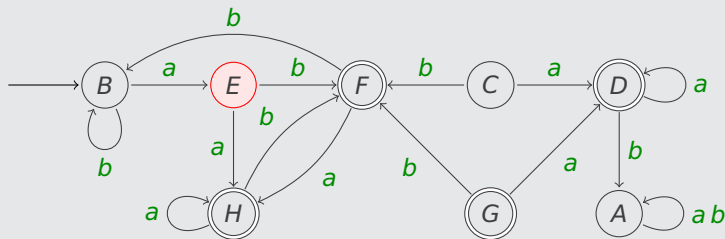
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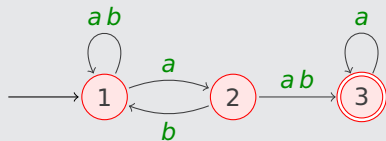
$D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



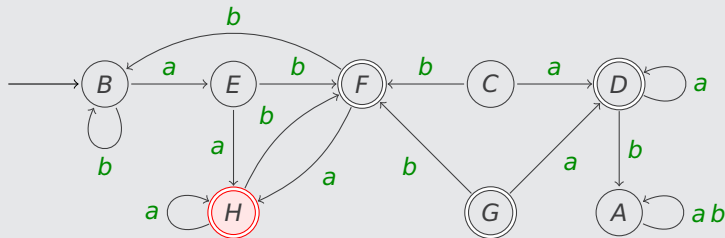
Example



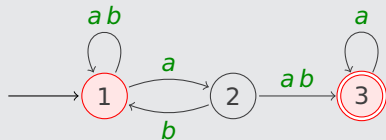
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	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



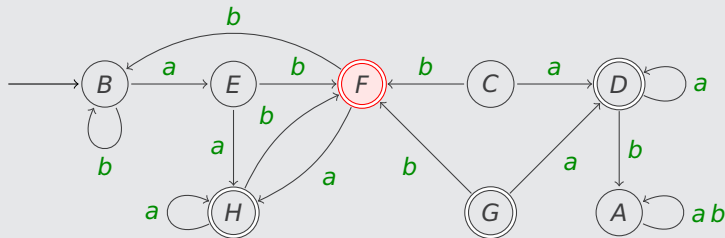
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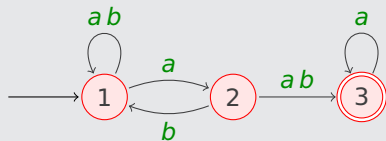
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	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



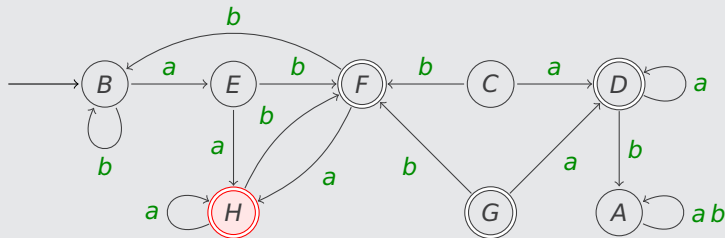
Example



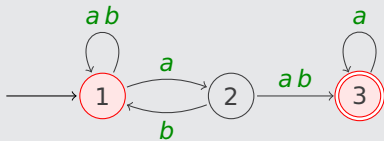
$A = \emptyset$ $E = \{1, 2\}$
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 $D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbbaababbabbbaaba**abba**



Example



$A = \emptyset$ $E = \{1, 2\}$

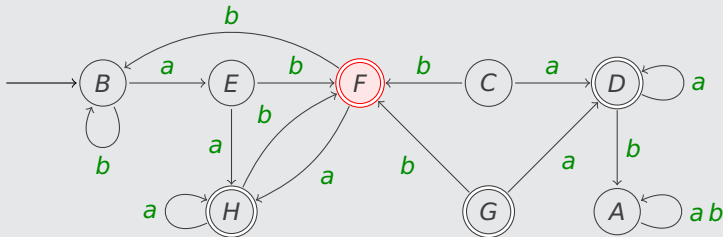
$B = \{1\}$ $F = \{1, 3\}$

$C = \{2\}$ $G = \{2, 3\}$

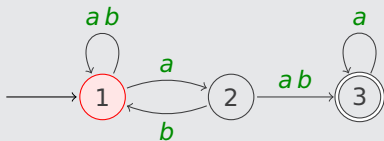
$D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbbaababbabbbaababba



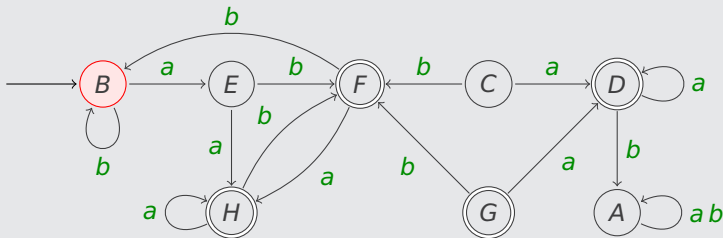
Example



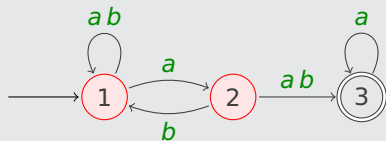
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 $B = \{1\}$ $F = \{1, 3\}$
 $C = \{2\}$ $G = \{2, 3\}$
 $D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



Example



$A = \emptyset$ $E = \{1, 2\}$

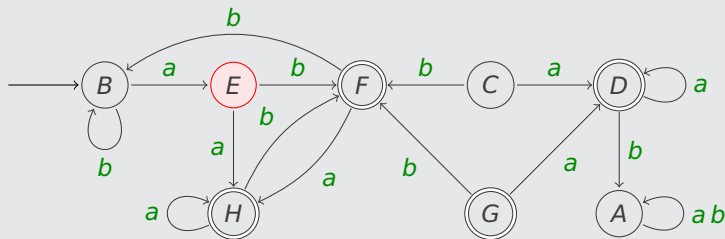
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$C = \{2\}$ $G = \{2, 3\}$

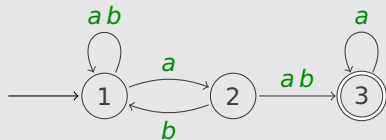
$D = \{3\}$ $H = \{1, 2, 3\}$

	a	b		a	b
A	A	A	E	H	F
B	E	B	F	H	B
C	D	F	G	D	F
D	D	A	H	H	F

abbbaababbabbbaababba



Example



abbbaababbabbbaababba

$A = \emptyset$ $E = \{1, 2\}$

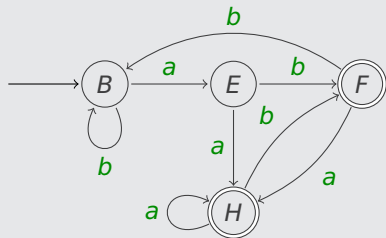
$B = \{1\}$ $F = \{1, 3\}$

$C = \{2\}$ $G = \{2, 3\}$

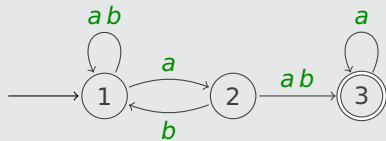
$D = \{3\}$ $H = \{1, 2, 3\}$

	<i>a</i>	<i>b</i>		<i>a</i>	<i>b</i>
<i>A</i>	<i>A</i>	<i>A</i>	<i>E</i>	<i>H</i>	<i>F</i>
<i>B</i>	<i>E</i>	<i>B</i>	<i>F</i>	<i>H</i>	<i>B</i>
<i>C</i>	<i>D</i>	<i>F</i>	<i>G</i>	<i>D</i>	<i>F</i>
<i>D</i>	<i>D</i>	<i>A</i>	<i>H</i>	<i>H</i>	<i>F</i>

remove inaccessible states



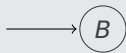
Example



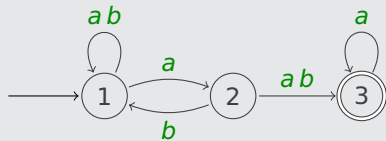
$$B = \{1\}$$

	<i>a</i>	<i>b</i>

	<i>a</i>	<i>b</i>



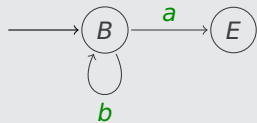
Example



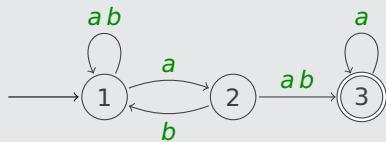
$$B = \{1\}$$

$$E = \{1, 2\}$$

	a	b		a	b
B	E	B			



Example

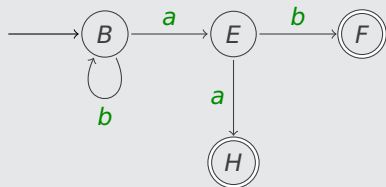


$$E = \{1, 2\}$$

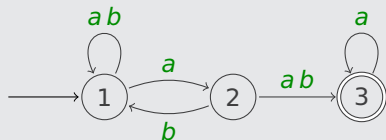
$$B = \{1\} \quad F = \{1, 3\}$$

$$H = \{1, 2, 3\}$$

	a	b		a	b
			E	H	F
B	E	B			



Example

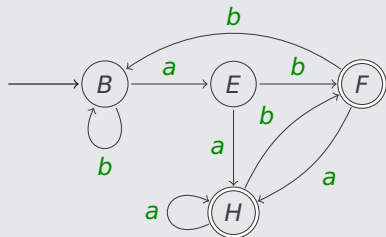


$$E = \{1, 2\}$$

$$B = \{1\} \quad F = \{1, 3\}$$

$$H = \{1, 2, 3\}$$

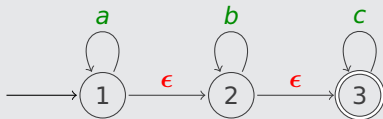
	a	b		a	b
			E	H	F
B	E	B	F	H	B
			H	H	F



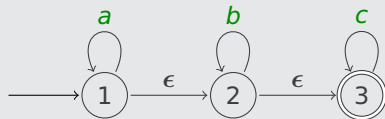
Outline

1. Summary of Previous Lecture
2. Nondeterministic Finite Automata
- 3. Epsilon Transitions**
4. Intermezzo
5. Closure Properties
6. Hamming Distance
7. Further Reading

Example



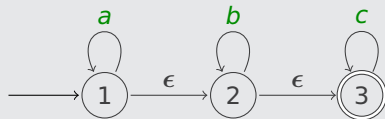
Example



Definitions

- **NFA with ϵ -transitions** (NFA_ϵ) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that

Example

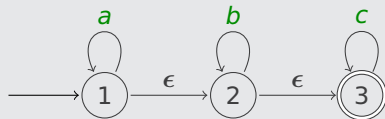


Definitions

► **NFA with ϵ -transitions** (NFA_ϵ) is sextuple $N = (Q, \Sigma, \epsilon, \Delta, S, F)$ such that

① $\epsilon \notin \Sigma$

Example

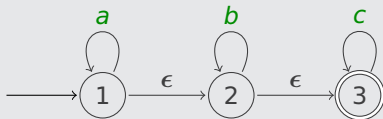


Definitions

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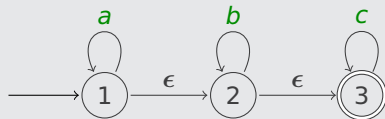
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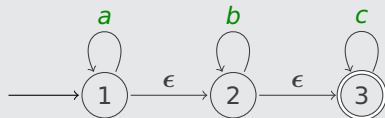


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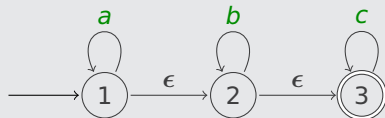
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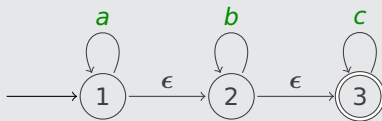
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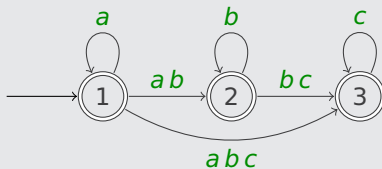
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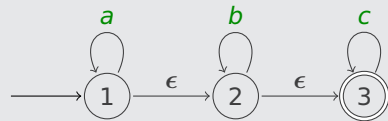
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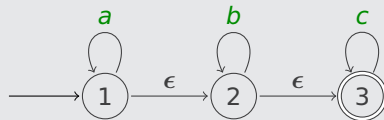
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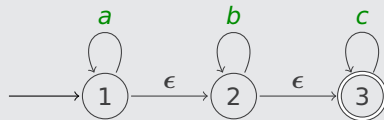


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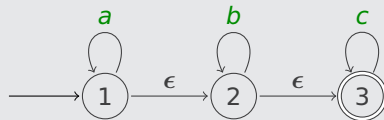
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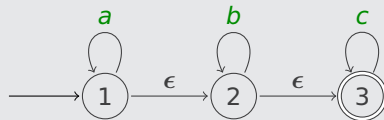
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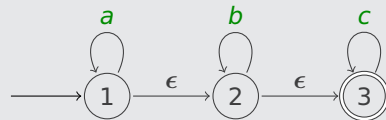
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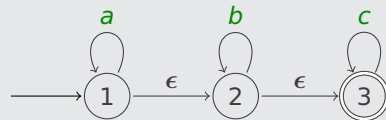
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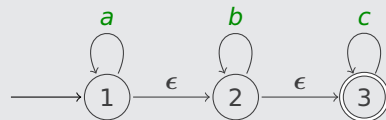
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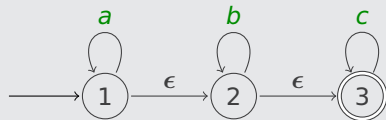
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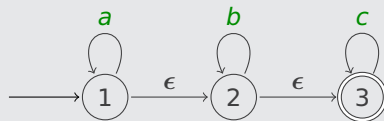
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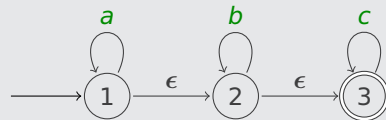
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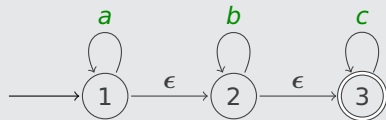
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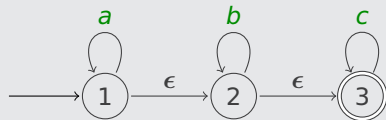
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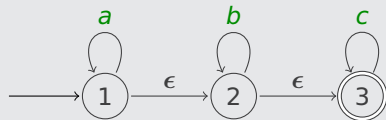
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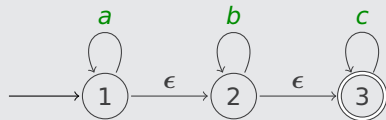
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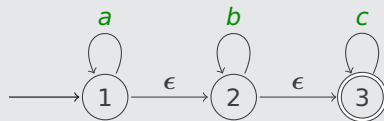
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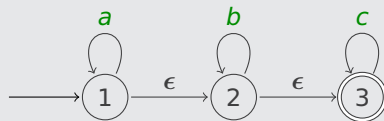
$$\begin{aligned} \Delta_2(3, b) &= \hat{\Delta}_1(\{3\}, b) = \bigcup \{C_\epsilon(\Delta_1(q, b)) \mid q \in \hat{\Delta}_1(\{3\}, \epsilon)\} \\ &= C_\epsilon(\Delta_1(3, b)) \end{aligned}$$

Example

NFA_ε $N_1 = (\{1, 2, 3\}, \{a, b, c\}, \epsilon, \Delta_1, \{1\}, \{3\})$ with

► Δ_1

	a	b	c	ϵ
1	$\{1\}$	$\emptyset$	$\emptyset$	$\{2\}$
2	$\emptyset$	$\{2\}$	$\emptyset$	$\{3\}$
3	$\emptyset$	$\emptyset$	$\{3\}$	$\emptyset$



NFA $N_2 = (\{1, 2, 3\}, \{a, b, c\}, \Delta_2, \{1\}, F_2)$ with

► $F_2 = \{q \mid C_\epsilon(\{q\}) \cap \{3\} \neq \emptyset\} = \{q \mid 3 \in C_\epsilon(\{q\})\} = \{1, 2, 3\}$

► Δ_2

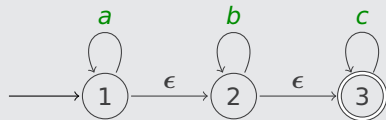
	a	b	c
1	$\{1, 2, 3\}$	$\{2, 3\}$	
2			
3			$\emptyset$

$$\begin{aligned} \Delta_2(3, b) &= \hat{\Delta}_1(\{3\}, b) = \bigcup \{C_\epsilon(\Delta_1(q, b)) \mid q \in \hat{\Delta}_1(\{3\}, \epsilon)\} \\ &= C_\epsilon(\quad \emptyset \quad) = \emptyset \end{aligned}$$

Example

$\text{NFA}_\epsilon N_1 = (\{1, 2, 3\}, \{a, b, c\}, \epsilon, \Delta_1, \{1\}, \{3\})$ with

Δ_1	a	b	c	ϵ
1	$\{1\}$	$\emptyset$	$\emptyset$	$\{2\}$
2	$\emptyset$	$\{2\}$	$\emptyset$	$\{3\}$
3	$\emptyset$	$\emptyset$	$\{3\}$	$\emptyset$



$\text{NFA } N_2 = (\{1, 2, 3\}, \{a, b, c\}, \Delta_2, \{1\}, F_2)$ with

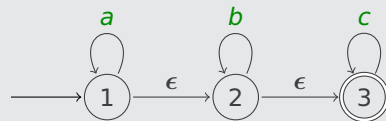
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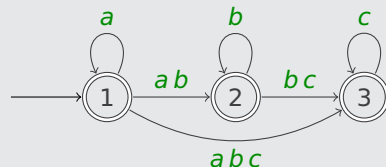
Δ_1	a	b	c	ϵ
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Outline

1. Summary of Previous Lecture
2. Nondeterministic Finite Automata
3. Epsilon Transitions
- 4. Intermezzo**
5. Closure Properties
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7. Further Reading

Question

What is the language accepted by the NFA_{ϵ} given by the following transition table ?

	ϵ	a	b
$\rightarrow 1$	$\emptyset$	$\{2\}$	$\emptyset$
2	$\{3\}$	$\emptyset$	$\emptyset$
3	$\emptyset$	$\{4\}$	$\{2\}$
$4 F$	$\{1\}$	$\emptyset$	$\emptyset$

- A** the set of all strings over $\{a, b\}$ starting and ending with a
- B** $\{xyx \mid x \in \{a\} \text{ and } y \in \{b\}^*\}$
- C** $\{\{a\}\{b\}^*\{a\}\}^+$
- D** $\{xyz \mid x, z \in \{a, b\} \text{ and } y \in \{aab\}^*\}$



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Theorem

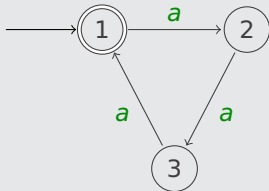
regular sets are **effectively** closed under **union**, **concatenation**, and **asterate**

Theorem

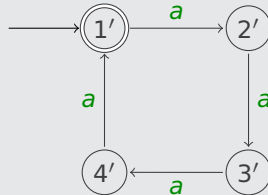
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$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\}$

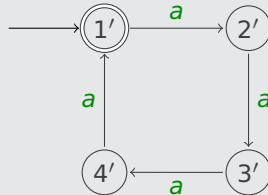
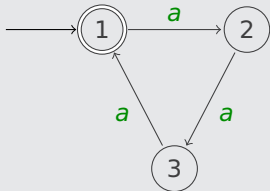


$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



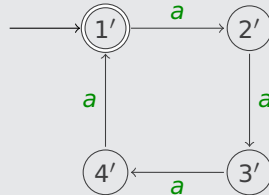
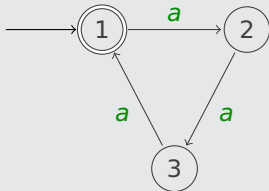
Example

$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\} \cup \{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$



Example

$$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3\} \cup \{x \in \{a\}^* \mid |x| \text{ is divisible by } 4\}$$



$$\{x \in \{a\}^* \mid |x| \text{ is divisible by } 3 \text{ or } 4\}$$

Theorem

regular sets are effectively closed under **union**, concatenation, and asterate

Proof

► $A = L(N_1)$ for NFA $N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$

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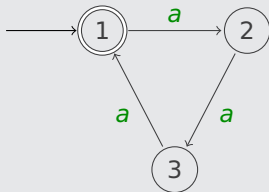
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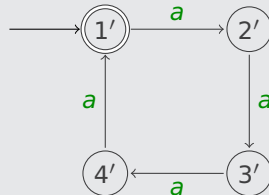
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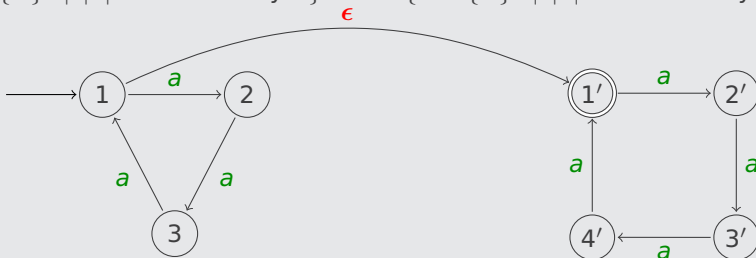
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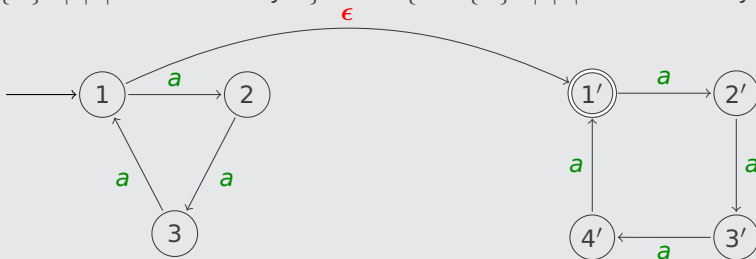
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$\{x \in \{a\}^* \mid |x| \notin \{1, 2, 5\}\}$

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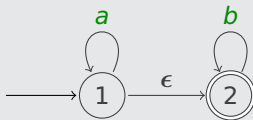
$$\textcircled{2} \quad \Delta(q, a) = \begin{cases} \Delta_1(q, a) & \text{if } q \in Q_1 \text{ and } a \in \Sigma \\ \Delta_2(q, a) & \text{if } q \in Q_2 \text{ and } a \in \Sigma \\ S_2 & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \emptyset & \text{otherwise} \end{cases}$$

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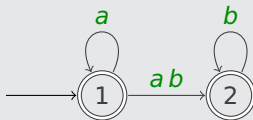
Example

$$\{a\}^* \{b\}^*$$



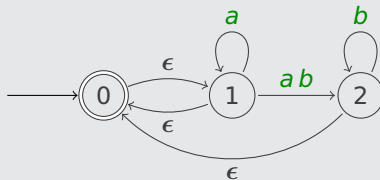
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$$\{a\}^* \{b\}^*$$



Example

$$(\{a\}^* \{b\}^*)^* = \{a, b\}^*$$



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Definitions

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► $A = L(M)$ for DFA $M = (Q_M, \{0, 1\}, \delta_M, s_M, F_M)$

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Proof (cont'd)

► key property:

$$(q, j) \in \widehat{\Delta}_N(\{(p, i)\}, y) \iff \widehat{\delta}_M(p, x) = q \text{ for some } x \in \{0, 1\}^*$$

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Proof (cont'd)

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Outline

1. Summary of Previous Lecture
2. Nondeterministic Finite Automata
3. Epsilon Transitions
4. Intermezzo
5. Closure Properties
6. Hamming Distance
- 7. Further Reading**

► Lecture 5 and 6

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Important Concepts

- ϵ -transition
- ϵ -closure
- asterate
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homework for October 23