



Automata and Logic

Aart Middeldorp and Johannes Niederhauser

Initial Remarks

- ▶ **Automata and Logic** is elective module 1 in master program Computer Science
- ▶ master students must select 3 out of 6 elective modules:
 - ① Automata and Logic
 - ② Constraint Solving (offered in 27S)
 - ③ Cryptography (offered in 27S)
 - ④ High-Performance Computing
 - ⑤ Optimisation and Numerical Computation
 - ⑥ Signal Processing and Algorithmic Geometry
- ▶ other master modules with theory content (**Logic and Learning** specialization):
 - ▶ Program and Resource Analysis (WM 8)
 - ▶ Computability Theory (WM 9)
 - ▶ Advanced Functional Programming (WM 20)
 - ▶ **The Lambda-Calculus, Typed and Untyped** (WM 20)

Outline

1. Introduction

Organisation

AI Policy

Contents

2. Basic Definitions

3. Deterministic Finite Automata

4. Intermezzo

5. Closure Properties

6. Further Reading

 with session ID **3017 2066** for anonymous questions



Important Information

- ▶ LVA 703302 (VO 2) + 703303 (PS 2)
- ▶ <https://cl-teaching.uibk.ac.at/26w/al>
- ▶ online registration for VO required
- ▶ OLAT links for VO and PS

Time and Place

VO	Monday	8:30–10:00	HSB 9	(AM)
PS	Friday	8:30–10:00	SR 12	(JN)

Consultation Hours

Aart Middeldorp	3M07	Monday	11:30–13:00
Johannes Niederhauser	3M03	Wednesday	10:00–11:30

Schedule

week 1	05.10 & 09.10	week 7	30.11 & 04.12
week 2	12.10 & 23.10	week 8	07.12 & 11.12
week 3	19.10 & 30.10 & 06.11	week 9	14.12 & 18.12 & 08.01
week 4	09.11 & 13.11	week 10	11.01 & 15.01
week 5	16.11 & 20.11	week 11	18.01 & 22.01
week 6	23.11 & 27.11	week 12	25.01

Grading — Vorlesung

- ▶ first exam on January 25
- ▶ registration starts 5 weeks and ends 2 weeks before exam
- ▶ de-registration is possible until 10:00 on January 22
- ▶ second exam on February 22; third exam on May 21 (on demand)

$\text{score} = \min\left(\frac{2}{3}(E + P) + B, 100\right)$

E : points for solved **exercises** (at most 130)
 P : points for **presentations** of solutions (at most 20)
 B : points for **bonus presentations**

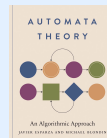
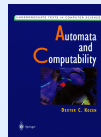
grade : $[0, 50) \rightarrow \mathbf{5}$ $[50, 63) \rightarrow \mathbf{4}$ $[63, 75) \rightarrow \mathbf{3}$ $[75, 88) \rightarrow \mathbf{2}$ $[88, 100] \rightarrow \mathbf{1}$

- ▶ homework exercises are given on course web site
- ▶ solved exercises must be marked and solutions must be uploaded (**PDF**) in **OLAT**
- ▶ strict deadline: 7 am on Friday
- ▶ 10 points per PS
- ▶ two presentations of solutions are mandatory
- ▶ 20 points for two presentations; additional presentations give bonus points
- ▶ attendance is compulsory; unexcused absence is allowed twice (resulting in 0 points)

evaluation 25W

Literature

- ▶ Dexter C Kozen
Automata and Computability
Springer-Verlag, 1997
- ▶ Javier Esparza and Michael Blondin
Automata Theory: An Algorithmic Approach
MIT Press, 2023
- ▶ Christel Baier and Joost-Pieter Katoen
Principles of Model Checking
MIT Press, 2008



Online Material

- ▶ solutions to selected exercises are available after they have been discussed in PS

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AI Policy — Automata and Logic (PS)

- ▶ students are responsible for all content they deliver (spoken, in writing, or code)
- ▶ **traffic light** system recognizes that policies differ between courses; its purpose is to provide students with certainty
- ▶ teachers have authority to redefine any aspect of policy and its enforcement
- ▶ failure to comply with policy will result in failing course



AI use is prohibited



AI use is allowed



AI use is permitted, but must be declared and comply with other restrictions (specific details to be provided by course teacher)



AI use is recommended

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Automata

- ▶ (**deterministic**, nondeterministic, alternating) **finite automata**
- ▶ regular expressions
- ▶ (alternating) Büchi automata

Logic

- ▶ (weak) monadic second-order logic
- ▶ Presburger arithmetic
- ▶ linear-time temporal logic

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- 2. Basic Definitions**
3. Deterministic Finite Automata
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Definitions

- ▶ **alphabet** is finite set; its elements are called **symbols** or **letters**
- ▶ **string** over alphabet Σ is finite sequence of elements of Σ
- ▶ **length** $|x|$ of string x is number of symbols in x
- ▶ **empty string** is unique string of length 0 and denoted by ϵ
- ▶ Σ^* is set of all strings over Σ ($\emptyset^* = \{\epsilon\}$)
- ▶ **language** over Σ is subset of Σ^*

Examples

strings over $\Sigma = \{0, 1\}$: 0 0110 ϵ

languages over Σ :

- ▶ $\{\epsilon, 0, 1, 00, 01, 10, 11\}$ (all strings having at most two symbols)
- ▶ $\{x \mid x \text{ is valid program in some machine language}\}$

Definitions

- ▶ **string concatenation** $x, y \in \Sigma^* \implies xy \in \Sigma^*$ is associative:

$$(xy)z = x(yz) \quad \text{for all } x, y, z \in \Sigma^*$$

- ▶ empty string is **identity** for concatenation:

$$\epsilon x = x\epsilon = x \quad \text{for all } x \in \Sigma^*$$

- ▶ x is **substring** (**prefix**, **suffix**) of y if $y = uxv$ ($y = xv$, $y = ux$)

- ▶ x^n ($x \in \Sigma^*$, $n \in \mathbb{N}$):

$$x^0 = \epsilon$$

$$x^{n+1} = x^n x$$

- ▶ $\#a(x)$ ($a \in \Sigma$, $x \in \Sigma^*$) denotes number of a 's in x

Definitions

for $A, B \subseteq \Sigma^*$

► **union**

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

► **intersection**

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

► **complement**

$$\sim A = \Sigma^* - A = \{x \in \Sigma^* \mid x \notin A\}$$

► **set concatenation**

$$AB = \{xy \mid x \in A \text{ and } y \in B\}$$

► **powers** A^n ($n \in \mathbb{N}$)

$$A^0 = \{\epsilon\} \quad A^{n+1} = AA^n$$

► **asterate** A^* is union of all finite powers of A

$$A^* = \bigcup_{n \geq 0} A^n = \{x_1 x_2 \cdots x_n \mid n \geq 0 \text{ and } x_i \in A \text{ for all } 1 \leq i \leq n\}$$

► $A^+ = AA^* = \bigcup_{n \geq 1} A^n$

► **power set**

$$2^A = \{Q \mid Q \subseteq A\}$$

Examples

- ▶ substrings of 011: 0, 1, 01, 11, 011, ϵ
- ▶ prefixes of 011: 0, 01, 011, ϵ
- ▶ suffixes of 011: 1, 11, 011, ϵ
- ▶ $(011)^3 = 011011011 \neq 011^3$
- ▶ $\#1(011011011) = 6$ $\#0(\epsilon) = 0$
- ▶ $\{0, 10, 111\}\{1, 11\} = \{01, 101, 1111, 011, 1011, 11111\}$
- ▶ $\{0, 01, 111\}\{1, 11\} = \{01, 011, 1111, 0111, 11111\}$
- ▶ $\{1, 01\}^3 = \{111, 0111, 1011, 01011, 1101, 01101, 10101, 010101\}$
- ▶ $\{1, 01\}^* = \{\epsilon, 1, 01, 11, 011, 101, 0101, 111, 0111, 1011, 01011, \dots\}$
- ▶ $2^{\{1, 01\}} = \{\emptyset, \{1\}, \{01\}, \{1, 01\}\}$

Some Useful Properties

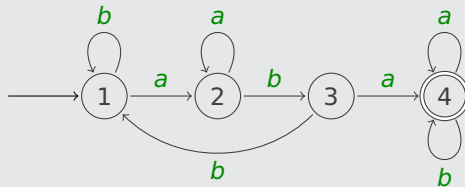
- ▶ $\{\epsilon\}A = A\{\epsilon\} = A$
- ▶ $\emptyset A = A\emptyset = \emptyset$
- ▶ $\sim(A \cup B) = (\sim A) \cap (\sim B)$
- ▶ $\sim(A \cap B) = (\sim A) \cup (\sim B)$
- ▶ $A^{m+n} = A^m A^n$
- ▶ $A^* A^* = A^*$
- ▶ $A^{**} = A^*$
- ▶ $A^* = \{\epsilon\} \cup AA^* = \{\epsilon\} \cup A^*A$
- ▶ $\emptyset^* = \{\epsilon\}$

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Example

DFA $M = (Q, \Sigma, \delta, s, F)$



1 $Q = \{1, 2, 3, 4\}$

2 $\Sigma = \{a, b\}$

3 $\delta: Q \times \Sigma \rightarrow Q$

4 $s = 1$

5 $F = \{4\}$

δ	a	b
1	2	1
2	2	3
3	4	1
4	4	4

$b \ a \ b \ a \ a \in L(M)$
1 1 2 3 4 4

$a \ a \ b \ b \ b \notin L(M)$
1 2 2 3 1 1

$L(M) = \{x \in \Sigma^* \mid x \text{ contains } aba \text{ as substring}\}$

Definitions

► **deterministic finite automaton (DFA)** is quintuple $M = (Q, \Sigma, \delta, s, F)$ with

- ① Q : finite set of **states**
- ② Σ : **input alphabet**
- ③ $\delta: Q \times \Sigma \rightarrow Q$: **transition function**
- ④ $s \in Q$: **start** state
- ⑤ $F \subseteq Q$: **final (accept) states**

► $\hat{\delta}: Q \times \Sigma^* \rightarrow Q$ is inductively defined by

$$\hat{\delta}(q, \epsilon) = q$$

$$\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$$

- string $x \in \Sigma^*$ is **accepted** by M if $\hat{\delta}(s, x) \in F$
- string $x \in \Sigma^*$ is **rejected** by M if $\hat{\delta}(s, x) \notin F$
- language accepted by M : $L(M) = \{x \in \Sigma^* \mid \hat{\delta}(s, x) \in F\}$
- set $A \subseteq \Sigma^*$ is **regular** if $A = L(M)$ for some DFA M

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Question

What is the language accepted by the DFA given by the following transition table ?

		<i>a</i>	<i>b</i>
→	1 <i>F</i>	2	1
	2 <i>F</i>	3	1
	3	3	3

Here the arrow indicates the start state and *F* marks the final states.

- A** $\{(ab)^n \mid n \in \mathbb{N}\}$
- B** $\sim(\{a, b\}^* \{aa\})$
- ☒ the set of all strings over $\{a, b\}$ not containing two consecutive *a*'s
- D** the set of all strings over $\{a, b\}$ with an odd number of *a*'s and an even number of *b*'s



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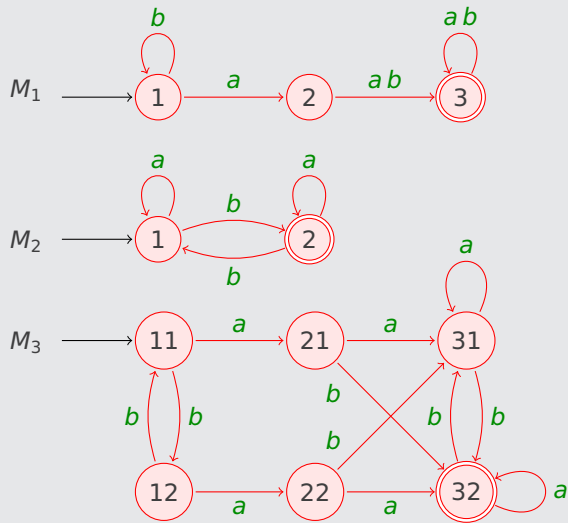
Theorem

regular sets are **effectively** closed under **intersection**

Proof (product construction)

- ▶ $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
 $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- ▶ $A \cap B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with
 - ① $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$
 - ② $F_3 = F_1 \times F_2$
 - ③ $s_3 = (s_1, s_2)$
 - ④ $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$ for all $p \in Q_1, q \in Q_2, a \in \Sigma$
- ▶ claim: $\widehat{\delta}_3((p, q), x) = (\widehat{\delta}_1(p, x), \widehat{\delta}_2(q, x))$ for all $x \in \Sigma^*$
proof of claim: easy induction on $|x|$ (on next slide)

Example



$$L(M_3) = L(M_1) \cap L(M_2)$$

Proof of Claim

claim: $\widehat{\delta}_3((p, q), x) = (\widehat{\delta}_1(p, x), \widehat{\delta}_2(q, x))$ for all $x \in \Sigma^*$

► base case: $|x| = 0$ and thus $x = \epsilon$

$$\widehat{\delta}_3((p, q), x) = (p, q) = (\widehat{\delta}_1(p, x), \widehat{\delta}_2(q, x))$$

► induction step: $|x| > 0$ and thus $x = ya$ with $|y| = |x| - 1$

$$\begin{aligned}\widehat{\delta}_3((p, q), x) &= \delta_3(\widehat{\delta}_3((p, q), y), a) && \text{(definition of } \widehat{\delta}_3\text{)} \\ &= \delta_3((\widehat{\delta}_1(p, y), \widehat{\delta}_2(q, y)), a) && \text{(induction hypothesis)} \\ &= (\delta_1(\widehat{\delta}_1(p, y), a), \delta_2(\widehat{\delta}_2(q, y), a)) && \text{(definition of } \delta_3\text{)} \\ &= (\widehat{\delta}_1(p, x), \widehat{\delta}_2(q, x)) && \text{(definition of } \widehat{\delta}_1 \text{ and } \widehat{\delta}_2\text{)}\end{aligned}$$

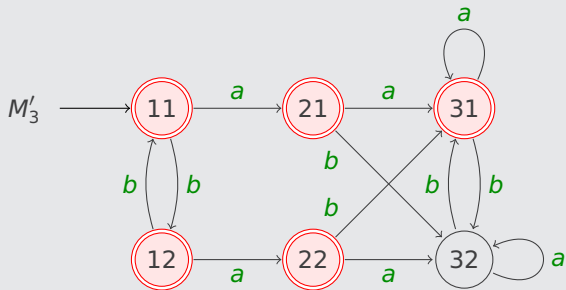
Theorem

regular sets are **effectively** closed under **complement**

Proof

- ▶ $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
- ▶ $\sim A = \Sigma^* - A = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ with
 - ① $Q_2 = Q_1$
 - ② $\delta_2(q, a) = \delta_1(q, a)$ for all $q \in Q_2$ and $a \in \Sigma$
 - ③ $s_2 = s_1$
 - ④ $F_2 = Q_1 - F_1$

Example



$$L(M'_3) = \sim L(M_3)$$

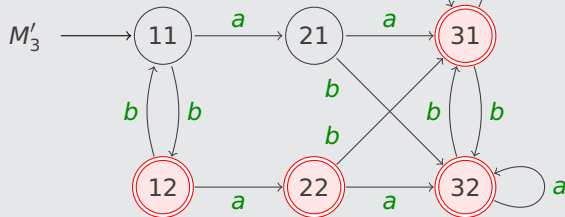
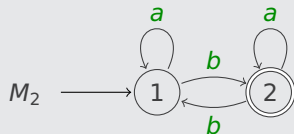
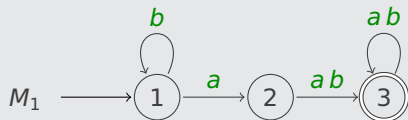
Theorem

regular sets are **effectively** closed under **union**

Proof (explicit construction)

- ▶ $A = L(M_1)$ for DFA $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$
 $B = L(M_2)$ for DFA $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$
- ▶ $A \cup B = L(M_3)$ for DFA $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$ with
 - ① $Q_3 = Q_1 \times Q_2 = \{(p, q) \mid p \in Q_1 \text{ and } q \in Q_2\}$
 - ② $F_3 = (F_1 \times Q_2) \cup (Q_1 \times F_2)$
 - ③ $s_3 = (s_1, s_2)$
 - ④ $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$ for all $p \in Q_1, q \in Q_2, a \in \Sigma$

Example



$$L(M'_3) = L(M_1) \cup L(M_2)$$

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- ▶ Lectures 1–4

Important Concepts

- ▶ alphabet
- ▶ closure properties
- ▶ DFA
- ▶ language
- ▶ product construction
- ▶ regular set
- ▶ string

homework for October 9

Solutions

- ... must be uploaded (PDF format) in OLAT **before 7 am on Friday**
- ... bonus exercises give **bonus points**