universität innsbruck

Automata and Logic	26W	LVA 703302 + 703303
Lecture 3		October 30 + November 6, 2026

Selected Solutions for October 30

2 We obtain

$$
\begin{array}{lll}
X_1 = aX_1 + aX_2 & X_2 = bX_1 + aX_3 & X_3 = aX_1 + bX_2 + aX_3 + \epsilon \\
X_1 = aX_1 + aX_2 & X_2 = bX_1 + aX_3 & X_3 = a^*(aX_1 + bX_2 + \epsilon) \qquad \text{(Arden's Lemma)} \\
X_1 = aX_1 + aX_2 & X_2 = bX_1 + aX_3 & X_3 = a^*aX_1 + a^*bX_2 + a^* \qquad \text{(distribute)} \\
X_1 = aX_1 + aX_2 & X_2 = bX_1 + a(a^*aX_1 + a^*bX_2 + a^*) \qquad \text{(substitute)} \\
X_1 = aX_1 + aX_2 & X_2 = (aa^*a + b)X_1 + aa^*bX_2 + aa^* \qquad \text{(distribute)} \\
X_1 = aX_1 + aX_2 & X_2 = (aa^*b)^*((aa^*a + b)X_1 + aa^*) \qquad \text{(Arden's Lemma)} \\
X_1 = aX_1 + aX_2 & X_2 = (aa^*b)^*(aa^*a + b)X_1 + (aa^*b)^*aa^* \qquad \text{(distribute)} \\
X_1 = aX_1 + a((aa^*b)^*(aa^*a + b)X_1 + (aa^*b)^*aa^*) \qquad \text{(substitute)} \\
X_1 = (a + a(aa^*b)^*(aa^*a + b))X_1 + a(aa^*b)^*aa^* \qquad \text{(distribute)} \\
X_1 = (a + a(aa^*b)^*(aa^*a + b))^*(a(aa^*b)^*aa^*) \qquad \text{(Arden's Lemma)}
\end{array}
$$

3 (a) String reversal is defined as follows:

$$\mathrm{rev}(a_1 \cdots a_n) = a_n \cdots a_1 \qquad \mathrm{rev}(A) = \{\mathrm{rev}(x) \mid x \in A\}$$

Let $A \subseteq \Sigma^*$ be a regular set. So $A = L(\alpha)$ for some regular expression α. We have $\mathrm{rev}(A) = L(\alpha')$ for the regular expression α' inductively defined as follows:

$$\alpha' = \begin{cases} a & \text{if } \alpha = a \in \Sigma \\ \epsilon & \text{if } \alpha = \epsilon \\ \varnothing & \text{if } \alpha = \varnothing \end{cases} \qquad \alpha' = \begin{cases} \beta' + \gamma' & \text{if } \alpha = \beta + \gamma \\ \gamma'\beta' & \text{if } \alpha = \beta\gamma \\ (\beta')^* & \text{if } \alpha = \beta^* \end{cases}$$

(b) Let A be an arbitrary regular set. From part (a) we know that $\mathrm{rev}(A)$ is regular. So $\mathrm{rev}(A) = L(M)$ for some DFA $M = (Q, \Sigma, \delta, s, F)$. By reversing the transitions and swapping start and final states, we obtain the NFA $N = (Q, \Sigma, \Delta, F, \{s\})$ with $\Delta(p, a) = \{q \mid \delta(q, a) = p\}$. One easily proves

$$\widehat{\delta}(p, x) \in X \iff p \in \widehat{\Delta}(X, \mathrm{rev}(x))$$

for all $p \in Q$, $X \subseteq Q$ and $x \in \Sigma^*$. Hence

$$\begin{array}{lllll} x \in L(N) & \iff & s \in \widehat{\Delta}(F, x) & \iff & \widehat{\delta}(s, \mathrm{rev}(x)) \in F \iff \mathrm{rev}(x) \in L(M) \\ & \iff & \mathrm{rev}(x) \in \mathrm{rev}(A) & \iff & x \in A \end{array}$$

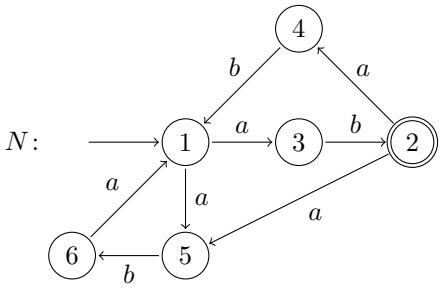
and thus $L(N) = A$. Note that N has exactly one final state.

Selected Solutions for November 6

2 (b) We proceed by induction on $|x|$. For $x = \epsilon$, we have $\widehat{\Delta}(\bigcup_{i\in I} A_i, \epsilon) = \bigcup_{i\in I} A_i = \bigcup_{i\in I} \widehat{\Delta}(A_i, \epsilon)$. For the step case, let $x = ya$ for some $y \in \Sigma^*$ and $a \in \Sigma$. We can reason as follows:

$$\begin{aligned}
\widehat{\Delta}(\bigcup_{i\in I} A_i, ya) &= \bigcup \{\, \Delta(q,a) \mid q \in \widehat{\Delta}(\bigcup_{i\in I} A_i, y) \,\} \\
&= \bigcup \{\, \Delta(q,a) \mid q \in \bigcup_{i\in I} \widehat{\Delta}(A_i, y) \,\} && \text{(induction hypothesis)} \\
&= \bigcup_{i\in I} \bigcup \{\, \Delta(q,a) \mid q \in \widehat{\Delta}(A_i, y) \,\} \\
&= \bigcup_{i\in I} \widehat{\Delta}(A_i, ya)
\end{aligned}$$

4 We obtain the NFA N



which satisfies $L(N) = \{h(x) \mid x \in L(M)\}$. The subset construction yields the equivalent DFA M'

		a	b
	A	A	A
$\rightarrow$	B	C	A
	C	A	D
	DF	E	A
	E	C	F
	F	G	A
	G	C	D

where

$A = \varnothing$ $\quad B = \{1\}$ $\quad C = \{3,5\}$ $\quad D = \{2,6\}$
$E = \{1,4,5\}$ $\quad F = \{1,6\}$ $\quad G = \{1,3,5\}$